CS21201 Discrete Structures

Tutorial 5

Pigeon-Hole Principle

1. You pick nine distinct points with integer coordinates in the three-dimensional space. Prove that there must exist two of these nine points—call them P and Q—such that the line segment PQ has a point on it (other than P and Q) with integer coordinates.

Solution Consider the parities (odd/even) of the coordinates of the points. For each point, there are eight possibilities. In the given collection of nine points, some possibility must be repeated. Take P and Q as the pair with the same parities in each of the coordinates. But then $(P+Q)/2$ is a point with integer coordinates.

2. Let $n \geqslant 10$ be an integer. You choose n distinct elements from the set $\{1,2,3,\ldots,n^2\}$. Prove that there must exist two disjoint subsets of the chosen numbers, whose sums are equal.

Solution The sum of the elements of a subset of $\{1,2,3,\ldots,n^2\}$ of size less than n is $< n^3$. The chosen collection has $2^n - 1$ non-empty subsets. For $n \geqslant 10$, we have $2^n - 1 > n^3$, so there must exist two different non-empty subsets A and B of the chosen numbers such that $\sum\limits_{a \in A} a = \sum\limits_{b \in B} b$. If A and B are not disjoint, take $A - (A \cap B)$ and $B - (A \cap B)$ as A and B.

3. Let ξ be an irrational number. Prove that given any real $\varepsilon > 0$ (no matter how small), there exist integers a, b such that $0 < a\xi - b < \varepsilon$.

Solution Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Choose an integer $n > \frac{1}{\varepsilon}$. Consider the $n+1$ fractional parts $\{\xi\}, \{2\xi\}, \{3\xi\}, \ldots, \{(n+1)\xi\}$. These are real numbers in the interval $[0,1)$. Break this interval into n non-empty sub-intervals $[0, \frac{1}{n}), [\frac{1}{n}, \frac{2}{n}), [\frac{2}{n}, \frac{3}{n}], \ldots, [\frac{n-1}{n}, 1)$. Two of the above $n+1$ fractional parts must belong to the same sub-interval. Call these fractional parts $\{i\xi\}$ and $\{j\xi\}$ with $\{i\xi\} \geqslant \{j\xi\}$. We have $i\xi = u + \{i\xi\}$ and $j\xi = v + \{j\xi\}$ for some integers u and v. But then $\{i\xi\} - \{j\xi\} = (i-j)\xi - (u-v)$ (so we take $a = i - j$ and $b = u - v$). By the choice of i and j, we have $\{i\xi\} - \{j\xi\} \leqslant \frac{1}{n} < \varepsilon$. If $\{i\xi\} - \{j\xi\} = 0$, we have $\xi = \frac{u-v}{i-j}$, which contradicts the fact that ξ is irrational. So we must have $\{i\xi\} - \{j\xi\} > 0$.

4. Let $p(x)$ be a polynomial with *integer* coefficients, having three distinct integer roots a, b, c. Prove that the polynomials $p(x) \pm 1$ cannot have any integer roots.

Solution Suppose that an integer d exists with $p(d) \pm 1 = 0$, that is, with $p(d) = \mp 1$. Clearly, d is different from a, b, c. For all $u, v \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, we have $(u-v) | (u^n - v^n)$, and so $(u-v) | (p(u) - p(v))$. But then, the non-zero differences $d-a, d-b, d-c$ all divide $\mp 1 - 0 = \mp 1$, and so can only be ± 1. Therefore, at least two of the three differences $d-a, d-b, d-c$ must be the same, contradicting the fact that a, b, c are distinct from one another.

5. 65 distinct integers are chosen in the range $1, 2, 3, \ldots, 2022$. Prove that there must exist four of the chosen integers (call them a, b, c, d) such that $a - b + c - d$ is a multiple of 2022.

Solution The total count of 2-subsets of the 65 chosen integers is $\binom{65}{2} = 2080 > 2022$. So we can find two distinct subsets $S = \{a, c\}$ and $T = \{b, d\}$ of the chosen integers such that $(a+c)$ rem $2022 = (b+d)$ rem 2022, that is, $(a-b+c-d)$ rem $2022 = 0$ (where rem means remainder of Euclidean division). We need to show that $S \cap T = \emptyset$. Suppose not. Since S and T are distinct, we must have $|S \cap T| = 1$. Say, $a = b$ (but $c \neq d$). But then, the condition $(a+c)$ rem $2022 = (b+d)$ rem 2022 implies that c rem $2022 = d$ rem 2022. But c and d are chosen in the range $[1, 2022]$, so they must be equal, a contradiction.

Additional Exercises

6. Let a be a positive odd integer. Prove that there exists a positive integer n such that $2^n - 1$ is divisible by a.

7. A repunit is an integer of the form $111\ldots1$. Prove that any $n \in \mathbb{N}$ with $\gcd(n, 10) = 1$ divides a repunit.

8. You pick six points in a 3×4 rectangle. Prove that two of these points must be at a distance $\leqslant \sqrt{5}$.

9. Let $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$. Use the pigeon-hole principle to prove that $ua + vb = 1$ for some $u, v \in \mathbb{Z}$.

10. Let ξ be an irrational number. Prove that given any real $\varepsilon > 0$ (no matter how small), there exist infinitely many pairs of integers a, b such that $0 < a\xi - b < \varepsilon$.

11. Show that there exists an integer n such that $0 < \sin n < 2^{-2022}$.

12. **(a)** Let p be a prime number, and x an integer not divisible by p. Prove that there exist non-zero integers a, b of absolute values less than $\sqrt{p}$ such that $p | (ax - b)$.

(b) Now assume that p is of the form $4k + 1$. We know from number theory that in this case there exists an integer x such that $p | (x^2 + 1)$. Show that $p = a^2 + b^2$ for some integers a, b.