CS21201 Discrete Structures

Tutorial 1

Elementary Counting Techniques

1. You are given r red balls, g green balls, and b blue balls. Assume that r, g, b are positive integers. Your task is to arrange the balls on a line subject to the following conditions. Find the count of all possible arrangements in each case.

(a) All blue balls appear together.

Solution Treat the b balls as a single object. The count is therefore $\dfrac{(r+g+1)!}{r!g!}$.

(b) The arrangement must start with a green ball and end with a non-green ball.

Solution Put a green ball at the beginning and a red/blue ball at the end. The count is therefore $\dfrac{(r-1+g-1+b)!}{(r-1)!(g-1)!b!} + \dfrac{(r+g-1+b-1)!}{r!(g-1)!(b-1)!} = \dfrac{(r+g+b-2)!}{(r-1)!(g-1)!(b-1)!}\left(\dfrac{1}{r}+\dfrac{1}{b}\right)$.

(c) No two red balls appear together.

Solution The g green balls and the b blue balls can be arranged in $\dfrac{(g+b)!}{g!b!}$ ways. Insert the r red balls in between or at the two ends without repetition. The count is therefore $\dfrac{(g+b)!}{g!b!}\dbinom{g+b+1}{r}$.

(d) No blue ball can appear after any red ball.

Solution Put all the b blue balls followed by all the r red balls. Insert the g green balls in between or at the two ends with repetitions allowed. The count is therefore $\dbinom{b+r+1+g-1}{g} = \dbinom{r+g+b}{g}$.

2. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of $n \geqslant 1$ real numbers. The arithmetic mean (average) of A is denoted by

$$x = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

A has $2^n - 1$ non-empty subsets $A_1, A_2, \ldots, A_{2^n-1}$. Let y_i denote the arithmetic mean of the elements of A_i. Finally, denote the arithmetic mean of these $2^n - 1$ arithmetic means as

$$y = \frac{y_1 + y_2 + \cdots + y_{2^n-1}}{2^n - 1}.$$

Prove that $x = y$.

Solution The contribution of each a_i in y is additive, so we can write

$$y = t_1 a_1 + t_2 a_2 + \cdots + t_n a_n.$$

Since this sum is symmetric about the n elements $a_1, a_2, \ldots, a_n$, we have $t_1 = t_2 = \cdots = t_n = t$ (say), and

$$y = t(a_1 + a_2 + \cdots + a_n).$$

It suffices to show that $t = \frac{1}{n}$.

Take any element of a of A (so $a = a_i$ for some i). Only those subsets of A, that contain a, leads to the contribution of a in the sum y. Choose some r in the range $[1, n]$. There are exactly $\binom{n-1}{r-1}$ subsets of A that are of size r and that contain a. For of each of these subsets, we divide the sum of the elements of the subset by r to obtain the arithmetic mean of the subset. It therefore follows that

$$t = \left[\frac{1}{1}\binom{n-1}{0} + \frac{1}{2}\binom{n-1}{1} + \frac{1}{3}\binom{n-1}{2} + \cdots + \frac{1}{n}\binom{n-1}{n-1}\right] \Big/ (2^n - 1).$$

By the binomial theorem, we have

$$(1+x)^{n-1} = \binom{n-1}{0} + \binom{n-1}{1}x + \binom{n-1}{2}x^2 + \cdots + \binom{n-1}{n-1}x^{n-1}.$$

Integrate (with respect to x) from $x = 0$ to $x = 1$ to obtain

$$\frac{2^n - 1}{n} = \left[\frac{1}{1}\binom{n-1}{0} + \frac{1}{2}\binom{n-1}{1} + \frac{1}{3}\binom{n-1}{2} + \cdots + \frac{1}{n}\binom{n-1}{n-1}\right].$$

3. How many sorted arrays of size n are there if each element of the array is an integer in the range $1, 2, 3, \ldots, r$?

Solution Let there be x_i occurrences of $i \in \{1, 2, 3, \ldots, r\}$. We have $x_1 + x_2 + x_3 + \cdots + x_r = n$ for $x_i \geqslant 0$. The answer is therefore $\binom{n+r-1}{n} = \binom{n+r-1}{r-1}$.

4. How many binary strings of length n are there with exactly k occurrences of the pattern 01? Assume that $n \geqslant 2k$.

Solution All such strings are of the form

1^*	0^*	01	1^*	0^*	01	1^*	0^*	01	1^*	0^*	...	1^*	0^*	01	1^*	0^*
x_1	x_2	2	x_3	x_4	2	x_5	x_6	2	x_7	x_8	...	x_{2k-1}	x_{2k}	2	x_{2k+1}	x_{2k+2}

That is, we want to count all non-negative integer solutions of

$$x_1 + x_2 + 2 + x_3 + x_4 + 2 + x_5 + x_6 + 2 + x_7 + x_8 + \cdots + x_{2k-1} + x_{2k} + 2 + x_{2k+1} + x_{2k+2} = n,$$

that is, of

$$x_1 + x_2 + \cdots + x_{2k+2} = n - 2k.$$

The answer is therefore $\binom{(n-2k)+(2k+2)-1}{2k+2-1} = \binom{n+1}{2k+1}$.

5. Prove the following identity for any positive integer n.

$$2^n = \binom{n+1}{1} + \binom{n+1}{3} + \binom{n+1}{5} + \cdots + \begin{cases} \binom{n+1}{n+1} & \text{if } n \text{ is even,} \\ \binom{n+1}{n} & \text{if } n \text{ is odd.} \end{cases}$$

Solution Vary k in the range $0, 1, 2, \ldots, \lfloor n/2 \rfloor$ in the last exercise. (This exercise can be solved using other methods, like the binomial theorem.)

6. Consider paths from $(0,0)$ to (n,n) in an $n \times n$ grid, that never cross the diagonal. Impose an additional constraint that these paths are not allowed to touch the main diagonal except only at the beginning and at the end. How many such constrained paths are there?

Solution $C(n-1)$.

7. Suppose that $m > n$. How many paths from $(0,0)$ to (m,n) with R and U movements are possible such that at no point of time, there are more U moves than R moves?

Solution Proceed as in the derivation of Catalan numbers. The answer is $\binom{m+n}{n} - \binom{m+n}{n-1} = \frac{m-n+1}{m+1}\binom{m+n}{n}$.

Additional Exercises

8. How many subsets of size k of $\{1,2,3,\ldots,n\}$ are there, that contain more odd numbers than even numbers?

9. **(a)** It is known that for all $i \geqslant 0$, the i-th Fibonacci number F_i is the integer closest to $\rho^i/\sqrt{5}$, where $\rho = (1+\sqrt{5})/2$ is the golden ratio. Assume that the math library calls `log` and `pow` take constant time per invocation. Propose an $O(1)$-time algorithm to determine how may Fibonacci numbers are there in the range $[1,n]$. Treat $F_1 = F_2 = 1$ as a single Fibonacci number.

(b) How many subsets of $\{1,2,3,\ldots,n\}$ contain exactly k Fibonacci numbers?

10. Take $A = \{a_1,a_2,\ldots,a_n\}$ and $x = (a_1+a_2+\cdots+a_n)/n$ as in Exercise 2. Fix an r in the range $1 \leqslant r \leqslant n$. There are $\binom{n}{r}$ subsets of A of size r. Let $z_{r,1}, z_{r,2}, \ldots, z_{r,\binom{n}{r}}$ denote the arithmetic means of these $\binom{n}{r}$ subsets. Take the arithmetic mean of these arithmetic means, that is, take

$$z_r = \frac{z_{r,1} + z_{r,2} + \cdots + z_{r,\binom{n}{r}}}{\binom{n}{r}}.$$

Finally, take the arithmetic mean of z_r, $r = 1,2,\ldots,n$, that is, take

$$z = \frac{z_1 + z_2 + \cdots + z_n}{n}.$$

Prove/Disprove: $z = x$.

* **11.** Let n be a positive integer. Expand n to the base 7 as

$$n = (d_l d_{l-1} d_{l-2} \ldots d_1 d_0)_7,$$

where each $d_i \in \{0,1,2,3,4,5,6\}$ is a 7-ary digit. Define

$$S_7(n) = d_l + d_{l-1} + d_{l-2} + \cdots + d_1 + d_0.$$

Notice that the sum $S_7(n)$ is not affected by leading zero digits. For example, the smallest 4-digit prime is

$$1009 = 2 \times 7^3 + 6 \times 7^2 + 4 \times 7 + 1 = (2641)_7 = (02641)_7 = (002641)_7 = \cdots,$$

and so

$$S_7(1009) = 2+6+4+1 = 13.$$

Prove/Disprove: If $p \neq 7$ is a prime, then $S_7(p)$ is a prime too. (**Hint:** Think modulo 6.)

12. In how many ways you can express 100 as a sum

$$a_1 + a_2 + a_3 + \cdots + a_r = 100$$

for some r with each $a_i \in \{1,2,3\}$ and with $a_1 \leqslant a_2 \leqslant a_3 \leqslant \cdots \leqslant a_r$.

13. How many paths from $(0,0)$ to (n,n) are possible with only R and U movements such that the paths never go above the line $y = x+1$?

14. How many paths from $(0,0)$ to (n,n) are possible with only R and U movements such that the paths lie entirely within the two lines $y = x-1$ and $y = x+1$? Touching these two lines is allowed.

15. Prove that the number of parenthesizations of the matrix product $A_0 A_1 A_2 \cdots A_n$ is equal to the n-th Catalan number $C(n)$.

16. Prove the following identity for all positive integers n:

$$\binom{2n}{n} = C(n) + \sum_{k=0}^{n-1} \binom{2n-2k-1}{n-k} C(k).$$