CS21201/CS21001 Discrete Structures, Autumn 2022–2023

Class Test 2

02–November–2022 | 06:30pm–07:30pm | Maximum marks: 20

Roll no: ____________ **Name:** ______________________________

[*Write your answers in the question paper itself. Be brief and precise. Answer all questions. If you use any algorithm/result/formula covered in the class, just mention it, do not elaborate.*]

1. Let $\mathbb{N}$ be the set of all positive integers. By constructing explicit injective maps, prove that the two sets

A = The set of all subsets of $\mathbb{N}$, and
B = The set of all subsets of $\mathbb{N}$ that do not contain consecutive integers

are equinumerous. Are these sets countable? Give a justification in only one sentence. **(6 + 2)**

Solution Since $B \subseteq A$, we have $|B| \leqslant |A|$ (consider the inclusion map). For proving that $|A| \leqslant |B|$, consider the injective map $A \to B$ that takes $S \subseteq \mathbb{N}$ to the subset $\{2n \mid n \in S\}$ of $\mathbb{N}$. Another possibility is mapping $\{a_1, a_2, a_3, \ldots\}$ with $a_1 < a_2 < a_3 < \cdots$ to $\{a_1, a_2 + 1, a_3 + 2, \ldots\}$.

These sets are not countable, because $\mathbb{N}$ (a countable set) cannot be equinumerous with its power set A.

2. A sequence $a_0, a_1, a_2, a_3, \ldots$ is defined recursively as

$$\begin{aligned} a_0 &= 1, \\ a_n &= a_{n-1} + 2a_{n-2} + 3a_{n-3} + \cdots + na_0 \text{ for } n \geqslant 1. \end{aligned}$$

(a) Derive a closed-form expression for the generating function $A(x)$ of this sequence. Show all the steps of your derivation. (**Hint:** Use convolution.) **(5)**

Solution We have

$$\begin{aligned} A(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \\ &= 1 + \sum_{n \geqslant 1} (a_{n-1} + 2a_{n-2} + 3a_{n-3} + \cdots + na_0)x^n \\ &= 1 + x(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)(1 + 2x + 3x^2 + 4x^3 + \cdots) \\ &= 1 + \frac{xA(x)}{(1-x)^2}. \end{aligned}$$

Simplification gives

$$A(x) = \frac{(1-x)^2}{1-3x+x^2}.$$

(b) From the generating function of Part (a), derive a closed-form formula for a_n. Show all the steps. **(5)**

Solution We have

$$A(x) = \frac{(1-x)^2}{1-3x+x^2} = 1 + \frac{x}{1-3x+x^2} = 1 + \frac{x}{(1-\alpha x)(1-\beta x)},$$

where $\alpha = \frac{3+\sqrt{5}}{2}$ and $\beta = \frac{3-\sqrt{5}}{2}$. Now, write

$$\frac{x}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x},$$

that is,

$$x = A(1-\beta x) + B(1-\alpha x).$$

Equating the constant term from both sides gives $A+B=0$, that is, $B=-A$. Then we equate the coefficient of x from both sides to get $1=(\alpha-\beta)A$. This gives $A=\frac{1}{\sqrt{5}}$ and $B=-\frac{1}{\sqrt{5}}$. We therefore have

$$a_n = \begin{cases} 1 & \text{if } n=0, \\ \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right] & \text{if } n \geqslant 1. \end{cases}$$

(c) From the formula of a_n derived in Part (b), deduce that $a_n = F_{2n}$ for all $n \geqslant 1$, where $F_0, F_1, F_2, \ldots$ is the Fibonacci sequence. (**Hint:** Use (without proving) the formula for Fibonacci numbers derived in the class.) **(2)**

Solution We have

$$F_{2n} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2n} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n}\right].$$

Finally, note that $\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$, and $\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{3-\sqrt{5}}{2}$.