CS21201/CS21001 Discrete Structures, Autumn 2022–2023

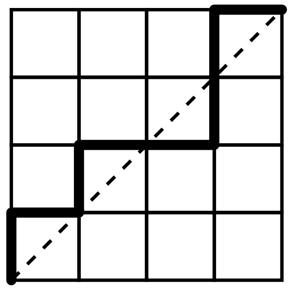
Class Test 1

07–September–2022 | 06:30pm–07:30pm | Maximum marks: 20

Roll no: ____________ **Name:** ______________________________

[*Write your answers in the question paper itself. Be brief and precise. Answer all questions. If you use any algorithm/result/formula covered in the class, just mention it, do not elaborate.*]

1. Consider paths in the grid from $(0,0)$ to (n,n) consisting of right and up movements only. Such a path is called diagonal-crossing if it crosses the main diagonal $y = x$ at least once. For example, the adjacent figure shows a path (for $n = 4$) that crosses the diagonal twice (horizontally at $(2,2)$ and vertically at $(3,3)$). Prove that the total number of diagonal-crossing paths from $(0,0)$ to (n,n) is $(n-1)C(n)$, where $C(n)$ is the n-th Catalan number. (**Hint:** First figure out the count of paths that are *not* diagonal-crossing.)



(6)

Solution The total number of paths from $(0,0)$ to (n,n) is $\binom{2n}{n}$. The count of paths that never go above the main diagonal is $C(n)$. If we reflect these paths about the line $y = x$, we get all the paths that never go below the main diagonal. All of the remaining paths are diagonal-crossing, so their count is

$$\binom{2n}{n} - 2C(n) = \binom{2n}{n} - \frac{2}{n+1}\binom{2n}{n} = \frac{n-1}{n+1}\binom{2n}{n} = (n-1)C(n).$$

2. Let p, q, r, s, t, u be propositions. Prove the validity of the following argument. Do not use truth tables. Use the rules of logical deduction instead. Show all the steps clearly. **(6)**

$$
\begin{array}{rl}
 & p \to (q \to r) \\
 & p \vee s \\
 & t \to q \\
 & \neg s \\
 & t \vee u \\
\hline
\therefore & r \vee u
\end{array}
$$

Solution We take $\neg r$ as an additional premise, and show that u follows. The steps are given below.

$$
\begin{array}{rll}
 & p \vee s & \text{Premise} \\
 & \neg s & \text{Premise} \\
\hline
\therefore & p & \text{Disjunctive syllogism}
\end{array}
\qquad
\begin{array}{rll}
 & p & \text{Derived} \\
 & p \to (q \to r) & \text{Premise} \\
\hline
\therefore & q \to r & \text{Modus ponens}
\end{array}
$$

$$
\begin{array}{rll}
 & t \to q & \text{Premise} \\
 & q \to r & \text{Derived} \\
\hline
\therefore & t \to r & \text{Syllogism}
\end{array}
\qquad
\begin{array}{rll}
 & t \to r & \text{Derived} \\
 & \neg r & \text{Additional premise} \\
\hline
\therefore & \neg t & \text{Modus tollens}
\end{array}
\qquad
\begin{array}{rll}
 & t \vee u & \text{Premise} \\
 & \neg t & \text{Derived} \\
\hline
\therefore & u & \text{Disjunctive syllogism}
\end{array}
$$

3. Let n be a positive integer. Consider all non-empty subsets of $\{1,2,3,\ldots,n\}$, that do not contain consecutive integers. Let S_n denote the sum of the squares of the products of the elements in these subsets. For example, for $n=5$, these subsets are

$$\{1\},\{2\},\{3\},\{4\},\{5\},\{1,3\},\{1,4\},\{1,5\},\{2,4\},\{2,5\},\{3,5\},\{1,3,5\}.$$

Therefore S_5 is equal to

$$1^2+2^2+3^2+4^2+5^2+(1\times3)^2+(1\times4)^2+(1\times5)^2+(2\times4)^2+(2\times5)^2+(3\times5)^2+(1\times3\times5)^2=719.$$

Prove that $S_n=(n+1)!-1$ for all $n\geqslant 1$.

(**Hint:** Categorize all non-empty subsets of $\{1,2,3,\ldots,n\}$ that contain n, and those that do not contain n.) **(8)**

Solution We proceed by (generalized weak) induction on n with $n_0=1$ and $k=2$.

[Basis] We need two basis cases. For $n=1$, we have $S_1=1^2=1$ and $(1+1)!-1=1$. For $n=2$, $S_2=1^2+2^2=5$ and $(2+1)!-1=6-1=5$.

[Induction] Assume that $S_{n-1}=n!-1$ and $S_{n-2}=(n-1)!-1$ for some $n\geqslant 3$. All non-empty subsets of $\{1,2,3,\ldots,n\}$ that do not contain consecutive integers can be classified in three groups.

1. Non-empty subsets of $\{1,2,3,\ldots,n-1\}$ that do not contain consecutive integers.
2. A non-empty subset with the desired property that contains n and one or more elements from $\{1,2,3,\ldots,n-1\}$. Since these subsets are not allowed to contain consecutive integers, the elements other than n must come from $\{1,2,3,\ldots,n-2\}$.
3. The subset $\{n\}$.

By induction, it follows that

$$\begin{aligned} S_n &= S_{n-1}+n^2S_{n-2}+n^2 \\ &= (n!-1)+n^2\Big((n-1)!-1\Big)+n^2 \\ &= n!+n^2\times(n-1)!-1 \\ &= (n-1)!(n+n^2)-1 \\ &= (n+1)!-1. \end{aligned}$$