Solving Linear Recurrences of Constant Orders

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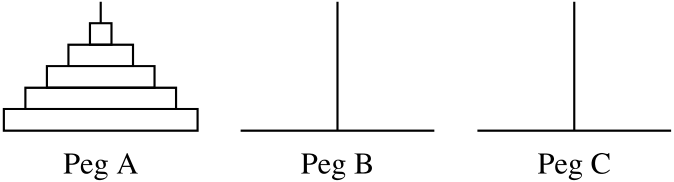
First-Order Recurrences

- Initial condition: A constant is supplied as a_0 or a_1.
- $a_n = c(n)a_{n-1} + f(n)$
- Unwind the recurrence

$$
\begin{aligned}
a_n &= c(n)a_{n-1} + f(n) \\
&= c(n)\Big(c(n-1)a_{n-2} + f(n-1)\Big) + f(n) \\
&= c(n)c(n-1)a_{n-2} + c(n)f(n-1) + f(n) \\
&= c(n)c(n-1)\Big(c(n-2)a_{n-3} + f(n-2)\Big) + [c(n)f(n-1) + f(n)] \\
&= c(n)c(n-1)c(n-2)a_{n-3} + [c(n)c(n-1)f(n-2) + c(n)f(n-1) + f(n)] \\
&\cdots
\end{aligned}
$$

- Eventually a_n is expressed in terms of the initial condition.

Tower of Hanoi



- Let n be the number of disks initially on Peg A.
- Let t_n be the minimum number of disk movements for n disks.
- $t_1 = 1$.
- $t_n = 2t_{n-1} + 1$ for $n \geqslant 2$.
- Unwind the recurrence:

$$\begin{aligned} t_n &= 2t_{n-1} + 1 = 2(2t_{n-2} + 1) + 1 = 2^2 t_{n-2} + (2+1) \\ &= 2^2(2t_{n-3} + 1) + (2+1) = 2^3 t_{n-3} + (2^2 + 2 + 1) = \cdots \\ &= 2^{n-1} t_1 + (2^{n-2} + \cdots + 2^2 + 2 + 1) = 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1 = 2^n - 1. \end{aligned}$$

Bubble sort

- Let c_n be the number of comparisons made for an array of size n.
- We have $c_1 = 0$, and $c_n = c_{n-1} + (n-1)$ for $n \geqslant 2$.
- Unwind the recurrence:

$$
\begin{aligned}
c_n &= c_{n-1} + (n-1) \\
&= c_{n-2} + (n-2) + (n-1) \\
&= c_{n-3} + (n-3) + (n-2) + (n-1) \\
&\cdots \\
&= c_1 + (1 + 2 + \cdots + (n-3) + (n-2) + (n-1)) \\
&= \frac{n(n-1)}{2}.
\end{aligned}
$$

Converting to the Homogeneous Form

- The recurrence $a_n = ca_{n-1}$ is easy to solve even if c is a non-constant function of n.
- Sometimes, we may convert a first-order recurrence to this form.

- Solve: $a_0 = 1$, and $a_n = na_{n-1} - n(n-2)$ for $n \geqslant 1$.
- We have $a_n = na_{n-1} - n^2 + 2n$, that is, $a_n - n = na_{n-1} - n(n-1) = n(a_{n-1} - (n-1))$.
- Let $b_n = a_n - n$ for all $n \geqslant 0$.
- We have $b_0 = 1$ and $b_n = nb_{n-1}$ for $n \geqslant 1$.
- Unwinding gives $b_n = n!$ for all $n \geqslant 0$.
- Therefore $a_n = n! + n$ for all $n \geqslant 0$.

Higher-Order Recurrences with Constant Coefficients

- Take a constant $k \in \mathbb{N}$, $k \geqslant 2$.
- General form: $c_k a_n + c_{k-1} a_{n-1} + \cdots + c_0 a_{n-k} = f(n)$.
- Here $c_k, c_{k-1}, \ldots, c_0$ are real-valued constants with $c_k \neq 0$ and $c_0 \neq 0$.
- k is the order of the recurrence.
- You need k initial conditions (like $a_0, a_1, \ldots, a_{k-1}$).
- Homogeneous recurrences: $f(n) = 0$.
- Non-homogeneous recurrences: $f(n) \neq 0$.
- The recurrence can be converted to the standard form

$$a_n = c_{k-1} a_{n-1} + c_{k-2} a_{n-2} + \cdots + c_0 a_{n-k} + f(n).$$

Unwinding does not help

- Fibonacci numbers: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$.
- $F_n = F_{n-1} + F_{n-2} = (F_{n-2} + F_{n-3}) + F_{n-2} = 2F_{n-2} + F_{n-3} = 2(F_{n-3} + F_{n-4}) + F_{n-3} = 3F_{n-3} + 2F_{n-4} = 3(F_{n-4} + F_{n-5}) + 2F_{n-4} = 5F_{n-4} + 3F_{n-5} = \cdots$.
- We have essentially established that $F_n = F_{k+1}F_{n-k} + F_kF_{n-k-1}$ for $k = 1, 2, 3, \ldots$.
- Put $k = n-1$ to get $F_n = F_nF_1 + F_{n-1}F_0 = F_n \times 1 + F_{n-1} \times 0 = F_n$.
- This is true, but useless.
- Generating functions can help.
- We have a standard procedure if
  - the coefficients $c_k, c_{k-1}, \ldots, c_0$ are constant, and
  - $f(n)$ is of some specific forms.
- The procedure is the discrete analog of solving differential equations.

The Characteristic Equation

- Homogeneous recurrence: $c_k a_n + c_{k-1} a_{n-1} + \cdots + c_0 a_{n-k} = 0$.
- The solution of $a_n = r a_{n-1}$ is of the form $a_n = A r^n$ for some constant A.
- We have $r \neq 0$. In general, we also have $A \neq 0$.
- Plugging in $a_n = A r^n$ gives $c_k A r^n + c_{k-1} A r^{n-1} + \cdots + c_0 A r^{n-k} = 0$.
- Since A and r are non-zero, this gives the characteristic equation

$$c_k r^k + c_{k-1} r^{k-1} + \cdots + c_0 = 0.$$

- For the homogeneous recurrence of the standard form $a_n = c_{k-1} a_{n-1} + c_{k-2} a_{n-2} + \cdots + c_0 a_{n-k}$, the characteristic equation is

$$r^k - c_{k-1} r^{k-1} - c_{k-2} r^{k-2} - \cdots - c_0 = 0.$$

- We need to factor the characteristic polynomial to get all its roots r.

Case of Unrepeated Roots

- The roots of the characteristic equation are real numbers and/or complex conjugate pairs.
- Assume that the k roots $r_1, r_2, \ldots, r_k$ are distinct from one another.
- The solution for the recurrence is given by

$$A_1 r_1^n + A_2 r_2^n + \cdots + A_k r_k^n$$

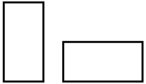
  for some constants $A_1, A_2, \ldots, A_k$.
- Plug in the k initial values to get a system of linear equations in $A_1, A_2, \ldots, A_k$.
- Solve the system to obtain these coefficients.

Fibonacci Numbers

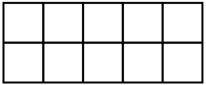
- $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$.
- The characteristic equation is $r^2 - r - 1 = 0$.
- The roots are $\rho = \dfrac{1+\sqrt{5}}{2}$ and $\bar{\rho} = \dfrac{1-\sqrt{5}}{2}$.
- We have $F_n = A\rho^n + B\bar{\rho}^n$ for all $n \geqslant 0$.
- The initial conditions give $A + B = 0$ and $A\rho + B\bar{\rho} = 1$.
- This linear system has the solution $A = \frac{1}{\sqrt{5}}$ and $B = -\frac{1}{\sqrt{5}}$.
- Therefore for all $n \geqslant 0$, we have

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] = \frac{\rho^n - \bar{\rho}^n}{\rho - \bar{\rho}}.$$

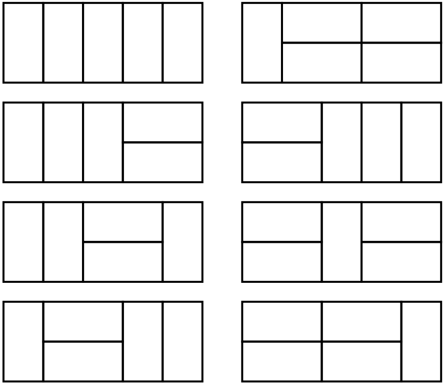
Covering a $2 \times n$ Board by Dominoes



Dominoes



Board to cover



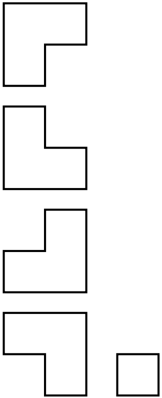
Possibilities

- $d_1 = 1$, $d_2 = 2$, $d_n = d_{n-1} + d_{n-2}$ for $n \geqslant 3$.
- $d_n = F_{n+1}$ for all $n \geqslant 1$.

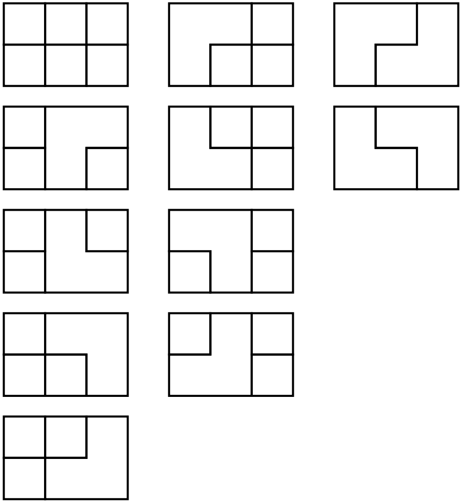
Palindromic Compositions of n

- $p_1 = 1$ and $p_2 = 2$.
- Take $n \geqslant 3$.
- A palindromic composition of n can be obtained from that of $n-2$ in two ways.
  - Add the summand 1 at the two ends.
  - Add 1 to the leftmost and the rightmost summands (may be the same).
- $p_n = 2p_{n-2}$ for $n \geqslant 3$. This is of order 2 with CE $r^2 - 2 = 0$, that is, $r = \pm\sqrt{2}$.
- The solution is $p_n = A(\sqrt{2})^n + B(\sqrt{2})^n$.
- $p_n = \left(\dfrac{\sqrt{2}+1}{2\sqrt{2}}\right)(\sqrt{2})^n + \left(\dfrac{\sqrt{2}-1}{2\sqrt{2}}\right)(-\sqrt{2})^n$ for all $n \geqslant 1$.
- This simplifies to $p_n = 2^{\lfloor n/2 \rfloor}$ for all $n \geqslant 1$.

Dominoes Again



Dominoes



Tiling

Dominoes Again: The Solution

- Initial conditions: $d_1 = 1$, $d_2 = 5$ and $d_3 = 11$.
- Recurrence: $d_n = d_{n-1} + 4d_{n-2} + 2d_{n-3}$ for $n \geqslant 4$.
- The characteristic equation $r^3 - r^2 - 4r - 2 = 0$ has roots $-1, 1+\sqrt{3}, 1-\sqrt{3}$.
- The initial conditions give

$$d_n = (-1)^n + \frac{1}{\sqrt{3}}(1+\sqrt{3})^n - \frac{1}{\sqrt{3}}(1-\sqrt{3})^n$$

for all $n \geqslant 1$.

Complex Roots

- Solve: $a_0 = 1$, $a_1 = 2$, $a_n = 2(a_{n-1} - a_{n-2})$ for $n \geqslant 2$.
- The characteristic equation is $r^2 - 2r + 2 = 0$, that is, $(r-1)^2 = -1$, that is, $r = 1 \pm \mathrm{i}$.
- We have $1 + \mathrm{i} = \sqrt{2}e^{\mathrm{i}\pi/4}$ and $1 - \mathrm{i} = \sqrt{2}e^{-\mathrm{i}\pi/4}$.
- Therefore $a_n = A(\sqrt{2})^n e^{\mathrm{i}n\pi/4} + B(\sqrt{2})^n e^{-\mathrm{i}n\pi/4}$.
- The initial conditions give $a_0 = 1 = A + B$ and $a_1 = 2 = A(1+\mathrm{i}) + B(1-\mathrm{i})$.
- Solving gives $A = \frac{1}{2}(1-\mathrm{i}) = \frac{1}{\sqrt{2}}e^{-\mathrm{i}\pi/4}$, and $B = \frac{1}{2}(1+\mathrm{i}) = \frac{1}{\sqrt{2}}e^{\mathrm{i}\pi/4}$.
- Therefore $a_n = (\sqrt{2})^{n-1}\left[e^{\mathrm{i}(n-1)\pi/4} + e^{-\mathrm{i}(n-1)\pi/4}\right] = (\sqrt{2})^{n+1}\cos\left[(n-1)\pi/4\right] = (\sqrt{2})^n\left[\cos(n\pi/4) + \sin(n\pi/4)\right]$.

Repeated Roots

- Let r be a root of the characteristic equation of multiplicity m.
- The contribution of r in the solution is

$$\left(A_{m-1}n^{m-1} + A_{m-2}n^{m-2} + \cdots + A_1 n + A_0\right) r^n,$$

  where $A_0, A_1, \ldots, A_{m-1}$ are constants.
- The solutions $n^i r^n$ are linearly independent for $i = 0, 1, 2, \ldots, m-1$.

Repeated Roots: Example

- Solve: $a_0 = 1$, $a_1 = 2$, $a_2 = 7$, $a_n = a_{n-1} + a_{n-2} - a_{n-3}$ for $n \geqslant 3$.
- The characteristic equation is $r^3 - r^2 - r + 1 = 0$, that is, $(r-1)^2(r+1) = 0$.
- The solution is of the form $a_n = (A_1 n + A_0) \times 1^n + B \times (-1)^n$.
- The initial conditions give $A_0 + B = 1$, $A_1 + A_0 - B = 2$, and $2A_1 + A_0 + B = 7$.
- The linear system has the solution $A_1 = 3$, $A_0 = 0$, and $B = 1$.
- We therefore have $a_n = 3n + (-1)^n$ for all $n \geqslant 0$.

Nonhomogeneous Recurrences

- $a_n = c_{k-1}a_{n-1} + c_{k-2}a_{n-2} + \cdots + c_0 a_{n-k} + f(n)$ with $c_0, c_1, \ldots, c_{k-1}$ constants and $f(n) \neq 0$.
- The recurrence is solvable for $f(n)$ of special forms.
- Procedure
  - Consider the homogeneous recurrence $a_n = c_{k-1}a_{n-1} + c_{k-2}a_{n-2} + \cdots + c_0 a_{n-k}$.
  - From the characteristic equation, determine the general form of the homogeneous solution $a_n^{(h)}$. Do not compute the coefficients in $a_n^{(h)}$ now.
  - Determine the general form of the particular solution $a_n^{(p)}$ from $f(n)$ and the characteristic equation.
  - Plug in the particular solution in the nonhomogeneous recurrence: $a_n^{(p)} = c_{k-1}a_{n-1}^{(p)} + c_{k-2}a_{n-2}^{(p)} + \cdots + c_0 a_{n-k}^{(p)} + f(n)$. This gives $a_n^{(p)}$ fully.
  - The solution to the nonhomogeneous recurrence is $a_n = a_n^{(h)} + a_n^{(p)}$ with $a_n^{(p)}$ known.
  - Plug in the initial conditions to determine (the coefficients in) $a_n^{(h)}$.

A Special Form of $f(n)$

- We consider $f(n) = p(n)s^n$, where
  - $p(n)$ is a non-zero polynomial of degree $t \geqslant 0$, and
  - s is a non-zero constant. We can have the invisible $s = 1$.
- Let m be the multiplicity of s as a root of the characteristic equation.
- If s is not a root of the characteristic equation, then $m = 0$.
- The particular solution is of the form

$$a_n^{(p)} = n^m \left(U_t n^t + U_{t-1} n^{t-1} + \cdots + U_1 n + U_0 \right) s^n,$$

  where $U_0, U_1, \ldots, U_t$ are constants.
- Put this solution in the given recurrence, and equate coefficients of n^i from both sides.
- This gives a linear system that determines $U_0, U_1, \ldots, U_t$.

Tower of Hanoi Revisited

- $t_1 = 1$, and $t_n = 2t_{n-1} + 1$ for $n \geqslant 2$.
- The characteristic equation is $r - 2 = 0$, that is, $r = 2$.
- The homogeneous solution is of the form $a_n^{(h)} = A \times 2^n$.
- The nonhomogeneous part is 1, so $p(n) = 1$ and $s = 1$ (s is not a root of the CE).
- We therefore have $t_n^{(p)} = n^0(U)1^n = U$.
- Put this in the recurrence to get $U = 2U + 1$, that is, $U = -1$, that is, $t_n^{(p)} = -1$.
- Therefore $t_n = t_n^{(h)} + t_n^{(p)} = A \times 2^n - 1$.
- Now use $t_1 = 1$ to get $1 = 2A - 1$, that is, $A = 1$.
- The final solution is $t_n = 2^n - 1$ for all $n \geqslant 1$.

Counting Sums in Recursive Fibonacci-Number Computation

- $s_0 = 0$, $s_1 = 0$, and $s_n = s_{n-1} + s_{n-2} + 1$ for $n \geqslant 2$.
- CE: $r^2 - r - 1 = 0$. The roots are $\rho = \frac{1+\sqrt{5}}{2}$ and $\bar{\rho} = \frac{1-\sqrt{5}}{2}$.
- $s_n^{(h)} = A\rho^n + B\bar{\rho}^n$.
- $s_n^{(p)} = U$, so $U = U + U + 1$, that is, $U = -1$, that is, $s_n = A\rho^n + B\bar{\rho}^n - 1$.
- The initial conditions give $s_0 = 0 = A + B - 1$ and $0 = A\rho + B\bar{\rho} - 1$.
- Solving gives $A = \dfrac{1+\sqrt{5}}{2\sqrt{5}}$ and $B = -\left(\dfrac{1-\sqrt{5}}{2\sqrt{5}}\right)$.
- Therefore the solution is

$$s_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right] - 1$$

for all $n \geqslant 0$. Indeed $s_n = F_{n+1} - 1$.

Nonhomogeneous Recurrence: Example

- Solve: $a_0 = 8$, $a_1 = 21$, and $a_n = a_{n-1} + 2a_{n-2} + 2n(2n+1)3^{n-2}$ for $n \geqslant 2$.
- CE: $r^2 - r - 2 = (r+1)(r-2) = 0$, that is, $r = -1, 2$, so $a_n^{(h)} = A(-1)^n + B2^n$.
- Particular solution: $p(n) = \frac{4}{9}n^2 + \frac{2}{9}n$ is of degree 2, and $s = 3$ is not a root of the CE), so $a_n^{(p)} = (U_2n^2 + U_1n + U_0)3^n$.
- Putting this in the recurrence and dividing by 3^{n-2}, we get $9\Big(U_2n^2 + U_1n + U_0\Big) = 3\Big(U_2(n-1)^2 + U_1(n-1) + U_0\Big) + 2\Big(U_2(n-2)^2 + U_1(n-2) + U_0\Big) + 2n(2n+1)$.
- Method of undetermined coefficients
  - Coefficient of n^2: $9U_2 = 3U_2 + 2U_2 + 4$, that is, $U_2 = 1$.
  - Coefficient of n: $9U_1 = -6U_2 + 3U_1 - 8U_2 + 2U_1 + 2$, that is, $U_1 = -3$.
  - Constant term: $9U_0 = 3U_2 - 3U_1 + 3U_0 + 8U_2 - 4U_1 + 2U_0$, that is, $U_0 = 8$.

Nonhomogeneous Recurrence: Example (Continued)

- $a_n^{(p)} = (n^2 - 3n + 8)3^n$.
- $a_n = a_n^{(h)} + a_n^{(p)} = A(-1)^n + B2^n + (n^2 - 3n + 8)3^n$.
- $a_0 = 8 = A + B + 8$, that is, $A + B = 0$.
- $a_1 = 21 = -A + 2B + 18$, that is, $-A + 2B = 3$.
- Solving this system, we get $A = -1$ and $B = 1$.
- Therefore for all $n \geqslant 0$, we have

$$a_n = 2^n - (-1)^n + (n^2 - 3n + 8)3^n.$$

Example: s is a Root of the Characteristic Equation

- Solve: $a_0 = 0$, $a_1 = 1$, and $a_n = a_{n-2} + n^2$ for $n \geqslant 2$.
- CE: $r^2 - 1 = 0$, that is, $r = \pm 1$, so $a_n^{(h)} = A + B(-1)^n$.
- $f(n) = n^2 = n^2 \times 1^n$, so $a_n^{(p)} = n(U_2 n^2 + U_1 n + U_0) \times 1^n = n(U_2 n^2 + U_1 n + U_0)$.
- Therefore $n(U_2 n^2 + U_1 n + U_0) = (n-2)(U_2 (n-2)^2 + U_1 (n-2) + U_0) + n^2$.
- Method of undetermined coefficients
  - Coefficient of n^3: $U_2 = U_2$ (no information).
  - Coefficient of n^2: $U_1 = -6U_2 + U_1 + 1$ which gives $U_2 = \frac{1}{6}$.
  - Coefficient of n: $U_0 = 12U_2 - 4U_1 + U_0$ which gives $U_1 = \frac{1}{2}$.
  - Constant term: $0 = -8U_2 + 4U_1 - U_0$ which gives $U_0 = \frac{1}{3}$.
- $a_n^{(p)} = \frac{1}{6} n(n^2 + 3n + 2)$, and so $a_n = A + B(-1)^n + \frac{1}{6} n(n^2 + 3n + 2)$.
- The initial conditions $a_0 = 0 = A + B$ and $a_1 = 1 = A - B + 1$ give $A = B = 0$.
- Therefore $a_n = \frac{1}{6} n(n^2 + 3n + 2)$ for all $n \geqslant 0$.