Advanced Counting Techniques

Generating Functions

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October 20, 2022

A Counting Problem

You appear in four tests.

- Algorithms
- Bioinformatics
- Compilers
- Discrete Structures

In each test, you get an integer mark in the range $[0, 10]$.

In how many ways can you get a total of 25 marks?

Some examples: $5+5+10+5 = 10+5+5+5 = 6+7+6+6 = 1+9+8+7 = 25$.

Frame the Problem Algebraically

- Algorithms: $A = 1 + a + a^2 + a^3 + a^4 + a^5 + a^6 + a^7 + a^8 + a^9 + a^{10}$.
- Bioinformatics: $B = 1 + b + b^2 + b^3 + b^4 + b^5 + b^6 + b^7 + b^8 + b^9 + b^{10}$.
- Compilers: $C = 1 + c + c^2 + c^3 + c^4 + c^5 + c^6 + c^7 + c^8 + c^9 + c^{10}$.
- Discrete Structures: $D = 1 + d + d^2 + d^3 + d^4 + d^5 + d^6 + d^7 + d^8 + d^9 + d^{10}$.

Consider the product $ABCD$.

The answer is the number of terms of the form $a^i b^j c^k d^l$ in $ABCD$ with $i + j + k + l = 25$.

No real progress actually.

An Insight

We are considering terms $a^i b^j c^k d^l$ with $i+j+k+l=25$.

We can take $a=b=c=d=x$.

The coefficient of x^{25} in

$$
\begin{aligned}
(1+x+x^2+x^3+\cdots+x^{10})^4 &= \left(\frac{1-x^{11}}{1-x}\right)^4 \\
&= \left(1-\binom{4}{1}x^{11}+\binom{4}{2}x^{22}-\binom{4}{3}x^{33}+x^{44}\right)\sum_{i\geqslant 0}\binom{i+3}{i}x^i
\end{aligned}
$$

gives the answer

$$
\binom{25+3}{25}-\binom{4}{1}\binom{14+3}{14}+\binom{4}{2}\binom{3+3}{3}=\binom{28}{25}-\binom{4}{1}\binom{17}{14}+\binom{4}{2}\binom{6}{3}=676.
$$

Exercise: Deduce the same formula by the principle of inclusion and exclusion.

Combination with Repetitions

To choose r objects with repetition from a set of n distinct objects.

Each object can be chosen a maximum of r times.

Look at the coefficient of x^r in $(1+x+x^2+\cdots+x^r)^n$.

To simplify matters, look at the infinite series

$$\begin{aligned}(1+x+x^2+\cdots)^n &= \left(\frac{1}{1-x}\right)^n \\ &= \frac{1}{(1-x)^n} \\ &= \sum_{i\geqslant 0}\binom{n+i-1}{i}x^i.\end{aligned}$$

The coefficient of x^r is $\binom{n+r-1}{r}$.

Definition

Let $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$ be an infinite sequence of real numbers.

The **generating function** of the sequence is

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots .$$

The power series $A(x)$ is *formal*.

We usually do not put any value for x in $A(x)$.

Consequently, the convergence of the series is usually not an issue.

If we want to put a value for x, convergence issues must be considered.

Examples

- Let $n \in \mathbb{N}$. Then $(1+x)^n$ is the generating function of

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}, 0, 0, 0, \ldots .$$

- Let $n \in \mathbb{N}$. Then $\dfrac{1-x^n}{1-x} = 1 + x + x^2 + \cdots + x^{n-1}$ is the generating function of

$$\underbrace{1, 1, 1, \ldots, 1,}_{n \text{ times}} 0, 0, 0, \ldots .$$

- $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots$ is the generating function of $1, 1, 1, \ldots$.

Examples

- $$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n + \cdots$$
  is the generating function of $1, 2, 3, 4, 5, \ldots$.

- $$\frac{x}{(1-x)^2} = 0 + x + 2x^2 + 3x^3 + 4x^4 + \cdots + nx^n + \cdots$$
  is the generating function of $0, 1, 2, 3, 4, 5, \ldots$.

- $$\frac{\mathrm{d}}{\mathrm{d}x} \frac{x}{(1-x)^2} = \frac{1+x}{(1-x)^3} = 1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots + (n+1)^2x^n + \cdots$$
  is the generating function of $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$.

- $\dfrac{x(1+x)}{(1-x)^3}$ is the generating function of $0^2, 1^2, 2^2, 3^2, 4^2, 5^2, \ldots$.

Examples

- $\dfrac{1}{1-\alpha x} = 1 + \alpha x + \alpha^2 x^2 + \alpha^3 x^3 + \cdots + \alpha^n x^n + \cdots$
  is the generating function of the geometric series $1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^n, \ldots$.
- If $A(x)$ is the generating function of $a_0, a_1, a_2, \ldots, a_n, \ldots$, and $B(x)$ the generating function of $b_0, b_1, b_2, \ldots, b_n, \ldots$, then $A(x) + B(x)$ is the generating function of $a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n, \ldots$.
- $A(x)B(x)$ is the generating function of the **convolution** $a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, \ldots, a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0, \ldots$.
- Take $B(x) = \dfrac{1}{1-x}$ in the convolution to see that $\dfrac{A(x)}{1-x}$ is the generating function of the **prefix sums** $a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots, a_0 + a_1 + a_2 + \cdots + a_n, \ldots$.

Examples

In how many ways 20 marbles can be placed in three boxes such that

(1) Each box contains at least two marbles, and
(2) The third box contains no more than ten marbles?

Look at the coefficient of x^{20} in

$$\begin{aligned} & (x^2+x^3+x^4+\cdots)^2(x^2+x^3+\cdots+x^{10}) \\ = \; & x^6(1+x+x^2+\cdots)^2(1+x+x^2+\cdots+x^8) \\ = \; & \frac{x^6(1-x^9)}{(1-x)^3} \\ = \; & (x^6-x^{15})\sum_{i\geqslant 0}\binom{i+2}{i}x^i. \end{aligned}$$

The answer is $\binom{14+2}{14}-\binom{5+2}{5}=\binom{16}{2}-\binom{7}{2}=120-21=99.$

Examples

How many 5-element subsets of $\{1,2,3,4,\ldots,20\}$ do not contain consecutive integers?

Let $\{a_1,a_2,a_3,a_4,a_5\}$ be such a subset with

$$1 = a_0 \leqslant a_1 < a_2 < a_3 < a_4 < a_5 \leqslant a_6 = 20.$$

For $i = 0,1,2,3,4,5$, define $d_i = a_{i+1} - a_i$.

We have $d_0, d_5 \geqslant 0$, $d_1,d_2,d_3,d_4 \geqslant 2$, and $d_0+d_1+d_2+d_3+d_4+d_5 = 20-1 = 19$.

The answer is the coefficient of x^{19} in

$$\begin{aligned} & (1+x+x^2+\cdots)^2(x^2+x^3+x^4+\cdots)^4 \\ = \; & \frac{x^8}{(1-x)^6} \; = \; x^8 \sum_{i \geqslant 0} \binom{i+5}{i} x^i, \end{aligned}$$

that is, $\binom{11+5}{11} = \binom{16}{5} = 4368$.

Geometric Distribution

- You toss a coin repeatedly until a head occurs.
- In each toss, p is the probability of head.
- Probability of tail is $q = 1 - p$ in each toss.
- Assume that $0 < p < 1$, so $0 < q < 1$ too.
- Let G be the number of times you need to toss.
- G assumes positive integral values.
- $\Pr[G = n] = q^{n-1}p$ for $n = 1, 2, 3, \ldots$.
- We want to compute $\mathrm{E}[G]$ and $\mathrm{Var}[G]$.

Expectation

- We have $\dfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$.
- The series converges for $|x| < 1$.
- Put $x = q$ to get

$$\frac{1}{(1-q)^2} = \frac{1}{p^2} = 1 + 2q + 3q^2 + 4q^3 + \cdots + nq^{n-1} + \cdots.$$

- $\mathrm{E}[G] = p + 2qp + 3q^2p + 4q^3p + \cdots + nq^{n-1}p + \cdots = p \times \dfrac{1}{p^2} = \dfrac{1}{p}$.

Variance

- $\mathrm{Var}(G) = \mathrm{E}[G^2] - \mathrm{E}[G]^2$.
- We have seen that $\dfrac{1+x}{(1-x)^3} = 1^2 + 2^2x + 3^2x^2 + \cdots + n^2x^{n-1} + \cdots$.
- This series too converges for $|x| < 1$.
- Put $x = q$ to get

$$1^2 + 2^2q + 3^2q^2 + \cdots + n^2q^{n-1} + \cdots = \frac{1+q}{(1-q)^3} = \frac{1+q}{p^3}.$$

- $\mathrm{E}[G^2] = 1^2p + 2^2qp + 3^2q^2p + 4^2q^3p + \cdots + n^2q^{n-1}p + \cdots = p \times \left(\dfrac{1+q}{p^3}\right) = \dfrac{1+q}{p^2}$.
- Thus $\mathrm{Var}(G) = \dfrac{1+q}{p^2} - \left(\dfrac{1}{p}\right)^2 = \dfrac{q}{p^2}$.

Compositions and Partitions of Positive Integers

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Ordered and Unordered Partitions of Positive Integers

Let $n \in \mathbb{N}$.

In how many ways can n be written as a sum of positive integers?

If the order of the summands is important, we talk about **compositions**.

If the order of the summands is not important, we talk about **partitions**.

Compositions of 4 are
$4 = 1+3 = 3+1 = 2+2 = 1+1+2 = 1+2+1 = 1+1+2 = 1+1+1+1.$

Partitions of 4 are $4 = 1+3 = 2+2 = 1+1+2 = 1+1+1+1.$

We proved earlier that the number of compositions of n is 2^{n-1}.

The number of partitions of n does not have a known closed-form formula.

We will study these again in the light of generating functions.

Counting Compositions of n

- Classify compositions by number of summands.
- One summand: Only one way of writing each $n \geqslant 1$. So the generating function is

$$x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{x}{1-x}.$$

- Two summands: Look at the coefficient of x^n in

$$(x + x^2 + x^3 + \cdots)^2 = \left(\frac{x}{1-x}\right)^2.$$

- In general, for i summands, consider the coefficient of x^n in

$$(x + x^2 + x^3 + \cdots)^i = \left(\frac{x}{1-x}\right)^i.$$

Counting Compositions of n

The generating function of the number of compositions of n is

$$\begin{aligned}
\sum_{i\geqslant 1}\left(\frac{x}{1-x}\right)^i &= \left(\frac{x}{1-x}\right)\sum_{i\geqslant 0}\left(\frac{x}{1-x}\right)^i \\
&= \left(\frac{x}{1-x}\right)\left[\frac{1}{1-\left(\frac{x}{1-x}\right)}\right] \\
&= \frac{x}{1-2x} \\
&= x(1+2x+2^2x^2+2^3x^3+\cdots+2^{n-1}x^{n-1}+\cdots) \\
&= x+2x^2+2^2x^3+2^3x^4+\cdots+2^{n-1}x^n+\cdots.
\end{aligned}$$

We have again derived that the number of compositions of n is 2^{n-1}.

Counting Palindromic Compositions of n

- $4 = 2+2 = 1+2+1 = 1+1+1+1.$
- $5 = 2+1+2 = 1+3+1 = 1+1+1+1+1.$
- If the number of summands is even, n must be even.
- If the number of summands is odd, then the middle summand must have the same parity as n.
- To the left of the center, any arbitrary composition is possible.
- To the right of the center, we write this composition in the reverse order.

Case 1: n is Odd

- n may be the only summand (one case).
- Now consider multiple summands.
- The number of summands must be odd.
- The central summand must be odd (any one of $1, 3, 5, 7, \ldots, n-2$).
- The remaining sum is $n-1, n-3, n-5, n-7, \ldots, 2$.
- This is distributed equally to the two sides of the center.
- The total count of palindromic compositions is therefore

$$1 + \left(2^{\frac{n-1}{2}-1} + 2^{\frac{n-3}{2}-1} + 2^{\frac{n-5}{2}-1} + \cdots + 2^{1-1}\right) = 2^{\frac{n-1}{2}} = 2^{\lfloor \frac{n}{2} \rfloor}.$$

Case 2: n is Even

- n may be the only summand (one case).
- First, consider odd number of summands.
- The central summand must be even (any one of $2, 4, 6, 8, \ldots, n-2$).
- The remaining sum is $n-2, n-4, n-6, n-8, \ldots, 2$.
- This is distributed equally to the two sides of the center.
- The total count of palindromic compositions with odd number of summands is $1 + \left(2^{\frac{n-2}{2}-1} + 2^{\frac{n-4}{2}-1} + 2^{\frac{n-6}{2}-1} + \cdots + 2^{1-1}\right) = 2^{\frac{n-2}{2}} = 2^{\frac{n}{2}-1}$.
- If the number of summands is even, any composition of $n/2$ gives a palindromic composition of n. The count in this case is $2^{\frac{n}{2}-1}$.
- The total count is $2^{\frac{n}{2}} = 2^{\lfloor \frac{n}{2} \rfloor}$.

Counting Partitions of n

- $p(n)$ is the number of partitions of n.
- We need to count how many times each $i \in \mathbb{N}$ may occur in the sum.
- 1 may occur never or once or twice or thrice or ... giving the power series $1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$.
- 2 may occur never or once or twice or thrice or ... giving the power series $1 + x^2 + x^4 + x^6 + \cdots = \dfrac{1}{1-x^2}$.
- In general, i may occur never or once or twice or thrice or ... giving the power series $1 + x^i + x^{2i} + x^{3i} + \cdots = \dfrac{1}{1-x^i}$.

Counting Partitions of n

- The generating function for $p(n)$ is

$$\prod_{i \geqslant 1} \frac{1}{1-x^i}.$$

- We may truncate the product after $i = n$.
- Nevertheless, we do not get any closed-form formula for $p(n)$.

Counting Partitions of n with Distinct Summands

- $p_d(n)$ is the number of partitions of n with distinct summands.
- $7 = 1+6 = 2+5 = 3+4 = 1+2+4$.
- $p_d(7) = 5$.
- Each $i \in \mathbb{N}$ occurs never or once in the sum.
- For any $i \geqslant 1$, the relevant series is $1 + x^i$.
- The generating function for $p_d(n)$ is therefore

$$(1+x)(1+x^2)(1+x^3)\cdots = \prod_{i \geqslant 1}(1+x^i).$$

Counting Partitions of n with Odd Summands

- $p_o(n)$ is the number of partitions of n with odd summands.
- $7 = 1+1+5 = 1+3+3 = 1+1+1+1+3 = 1+1+1+1+1+1+1.$
- $p_o(7) = 5.$
- Only $1, 3, 5, 7, \ldots$ are allowed in the sum.
- The generating function for $p_o(n)$ is therefore

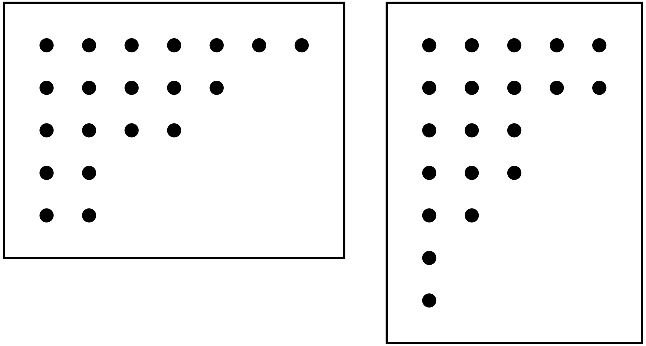
$$
\begin{aligned}
& (1+x+x^2+x^3+\cdots)(1+x^3+x^6+x^9+\cdots)(1+x^5+x^{10}+x^{15}+\cdots)\cdots \\
= \;& \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^5}\right)\cdots \\
= \;& \prod_{i\geqslant 1}\left(\frac{1}{1-x^{2i-1}}\right).
\end{aligned}
$$

Equality of the Last Two Generating Functions

Use $1+x^i = \dfrac{1-x^{2i}}{1-x^i}$.

$$
\begin{aligned}
\prod_{i\geqslant 1}(1+x^i) &= \prod_{i\geqslant 1}\frac{1-x^{2i}}{1-x^i}\\
&= \left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^4}{1-x^2}\right)\left(\frac{1-x^6}{1-x^3}\right)\left(\frac{1-x^8}{1-x^4}\right)\cdots\\
&= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^5}\right)\cdots.
\end{aligned}
$$

It follows that $p_d(n) = p_o(n)$ for all $n \in \mathbb{N}$.



$$20 = 7+5+4+2+2 = 5+5+3+3+2+1+1.$$

Observation: The number of partitions of n into m summands is equal to the number of partitions of n into summands, among which m is the largest.

Exercise: Prove that the number of partitions of n is equal to the number of partitions of $2n$ into n summands.

Using Generating Functions

to Solve Recurrence Relations

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October 20, 2022

A Simple Recurrence

$$\begin{aligned} a_0 &= 1 \\ a_n &= 3a_{n-1} + 2n \ \text{ for } \ n \geqslant 1. \end{aligned}$$

The first few terms of the sequence are

$$\begin{aligned} a_0 &= 1, \\ a_1 &= 3 \times 1 + 2 \times 1 = 5, \\ a_2 &= 3 \times 5 + 2 \times 2 = 19, \\ a_3 &= 3 \times 19 + 2 \times 3 = 63, \\ a_4 &= 3 \times 63 + 2 \times 4 = 197, \\ &\ldots \end{aligned}$$

The Generating Function of the Sequence

$$
\begin{aligned}
A(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \\
&= 1 + (3a_0 + 2 \times 1)x + (3a_1 + 2 \times 2)x^2 + (3a_2 + 2 \times 3)x^3 + \cdots + (3a_{n-1} + 2n)x^n + \cdots \\
&= 1 + 3x(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots) + \\
&\qquad 2x(1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots) \\
&= 1 + 3xA(x) + \frac{2x}{(1-x)^2}.
\end{aligned}
$$

Therefore

$$A(x) = \frac{1}{1-3x} + \frac{2x}{(1-x)^2(1-3x)}.$$

Expand $A(x)$

$$A(x) = \frac{1}{1-3x} + \frac{2x}{(1-x)^2(1-3x)} = \frac{A}{1-3x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}.$$

$$(1-x)^2 + 2x = A(1-x)^2 + B(1-x)(1-3x) + C(1-3x).$$

Put $x = \frac{1}{3}$ to get $A = \dfrac{(1-\frac{1}{3})^2 + \frac{2}{3}}{(1-\frac{1}{3})^2} = \dfrac{4+6}{4} = \dfrac{5}{2}$.

Put $x = 1$ to get $C = \dfrac{(1-1)^2+2}{1-3} = -1$.

Equate the constant coefficient to get $1 = A+B+C$, so $B = 1-A-C = 1-\frac{5}{2}+1 = -\frac{1}{2}$.

Power Series Expansion of $A(x)$

$$
\begin{aligned}
A(x) &= \frac{\frac{5}{2}}{1-3x} - \frac{\frac{1}{2}}{1-x} - \frac{1}{(1-x)^2} \\
&= \frac{5}{2}\left[1 + 3x + 3^2x^2 + 3^3x^3 + \cdots + 3^nx^n + \cdots\right] \\
&\quad - \frac{1}{2}\left[1 + x + x^2 + x^3 + \cdots + x^n + \cdots\right] \\
&\quad - \left[1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n + \cdots\right].
\end{aligned}
$$

Therefore $a_n = \frac{5}{2} \times 3^n - \frac{1}{2} - (n+1) = \frac{5}{2} \times 3^n - n - \frac{3}{2}$ for all $n \geqslant 0$.

Fibonacci Numbers

$$
\begin{aligned}
F_0 &= 0, \\
F_1 &= 1, \\
F_n &= F_{n-1} + F_{n-2} \text{ for } n \geqslant 2.
\end{aligned}
$$

The generating function of the Fibonacci sequence is

$$F(x) = F_0 + F_1 x + F_2 x^2 + F_3 x^3 + \cdots + F_n x^n + \cdots.$$

Derivation of $F(x)$

$$
\begin{aligned}
F(x) &= F_0 + F_1 x + F_2 x^2 + F_3 x^3 + \cdots + F_n x^n + \cdots \\
&= 0 + x + (F_1 + F_0)x^2 + (F_2 + F_1)x^3 + (F_3 + F_2)x^4 + \cdots + (F_{n-1} + F_{n-2})x^n + \cdots \\
&= x + (F_1 x^2 + F_2 x^3 + F_3 x^4 + \cdots + F_{n-1} x^n + \cdots) + \\
&\qquad (F_0 x^2 + F_1 x^3 + F_2 x^4 + \cdots + F_{n-2} x^n + \cdots) \\
&= x - F_0 x + (F_0 x + F_1 x^2 + F_2 x^3 + F_3 x^4 + \cdots + F_{n-1} x^n + \cdots) + \\
&\qquad\qquad (F_0 x^2 + F_1 x^3 + F_2 x^4 + \cdots + F_{n-2} x^n + \cdots) \\
&= x + x(F_0 + F_1 x + F_2 x^2 + F_3 x^3 + \cdots + F_n x^n + \cdots) + \\
&\qquad x^2 (F_0 + F_1 x + F_2 x^2 + F_3 x^3 + \cdots + F_n x^n + \cdots) \\
&= x + xF(x) + x^2 F(x).
\end{aligned}
$$

Therefore

$$F(x) = \frac{x}{1 - x - x^2}.$$

Playing with $F(x)$

We want to write the denominator as

$$1 - x - x^2 = (1 - \alpha x)(1 - \beta x).$$

Therefore we solve the quadratic equation $y^2 - y - 1 = 0$.

The roots are $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$.

This gives $F(x) = \dfrac{x}{1-x-x^2} = \dfrac{x}{(1-\alpha x)(1-\beta x)} = \dfrac{A}{1-\alpha x} + \dfrac{B}{1-\beta x}$.

It follows that $x = A(1-\beta x) + B(1-\alpha x)$.

Put $x = 1/\alpha$ and $x = 1/\beta$ to get

$$A = \frac{1/\alpha}{1-\beta/\alpha} = \frac{1}{\alpha-\beta} = \frac{1}{\sqrt{5}} \quad \text{and} \quad B = \frac{1/\beta}{1-\alpha/\beta} = \frac{1}{\beta-\alpha} = -\frac{1}{\sqrt{5}}.$$

Explicit Formula for Fibonacci Numbers

We have derived

$$F(x) = \frac{1}{\sqrt{5}} \left[\frac{1}{1-\alpha x} - \frac{1}{1-\beta x} \right].$$

Power-series expansions of the two terms give

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5}} \left(\alpha^n - \beta^n \right) \\ &= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] \end{aligned}$$

for all $n \geqslant 0$.

A More Complicated Recurrence

$$\begin{aligned} b_0 &= 1, \\ b_1 &= 2, \\ b_2 &= 3, \\ b_n &= b_{n-1} + b_{n-2} - b_{n-3} + 4 \text{ for all } n \geqslant 3. \end{aligned}$$

A few other terms of the sequence are

$$\begin{aligned} b_3 &= 3+2-1+4 = 8, \\ b_4 &= 8+3-2+4 = 13, \\ b_5 &= 13+8-3+4 = 22, \\ b_6 &= 22+13-8+4 = 31, \\ &\cdots \end{aligned}$$

The Generating Function of the Sequence

$$
\begin{aligned}
B(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots + b_n x^n + \cdots \\
&= 1 + 2x + 3x^2 + \sum_{n \geqslant 3} (b_{n-1} + b_{n-2} - b_{n-3} + 4) x^n \\
&= 1 + 2x + 3x^2 + x \sum_{n \geqslant 3} b_{n-1} x^{n-1} + x^2 \sum_{n \geqslant 3} b_{n-2} x^{n-2} - x^3 \sum_{n \geqslant 3} b_{n-3} x^{n-3} + 4x^3 \sum_{n \geqslant 3} x^{n-3} \\
&= 1 + 2x + 3x^2 + x\Big(B(x) - b_0 - b_1 x\Big) + x^2\Big(B(x) - b_0\Big) - x^3 B(x) + \frac{4x^3}{1-x} \\
&= 1 + 2x + 3x^2 - x - 2x^2 - x^2 + \Big(x + x^2 - x^3\Big) B(x) + \frac{4x^3}{1-x} \\
&= 1 + x + \Big(x + x^2 - x^3\Big) B(x) + \frac{4x^3}{1-x}.
\end{aligned}
$$

Therefore $B(x) = \dfrac{1 - x^2 + 4x^3}{(1-x)(1-x-x^2+x^3)} = \dfrac{1 - x^2 + 4x^3}{(1-x)^3(1+x)}$.

Expand $B(x)$

$$B(x) = \frac{1-x^2+4x^3}{(1-x)^3(1+x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D}{1+x}.$$

$$1-x^2+4x^3 = A(1-x)^2(1+x) + B(1-x)(1+x) + C(1+x) + D(1-x)^3.$$

Put $x = 1$ to get $C = 2$.

Put $x = -1$ to get $D = -\frac{1}{2}$.

Coefficient of x^3: $4 = A - D$, so $A = 4 + D = \frac{7}{2}$.

Constant term: $1 = A + B + C + D$, so $B = 1 - A - C - D = -4$.

Power Series Expansion of $B(x)$

$$\begin{aligned} B(x) &= \frac{\frac{7}{2}}{1-x} - \frac{4}{(1-x)^2} + \frac{2}{(1-x)^3} - \frac{\frac{1}{2}}{1+x} \\ &= \frac{7}{2}\sum_{n\geqslant 0} x^n - 4\sum_{n\geqslant 0}(n+1)x^n + 2\sum_{n\geqslant 0}\frac{(n+1)(n+2)}{2!}x^n - \frac{1}{2}\sum_{n\geqslant 0}(-1)^n x^n. \end{aligned}$$

Therefore

$$\begin{aligned} b_n &= \frac{7}{2} - 4(n+1) + (n+1)(n+2) - \frac{1}{2}(-1)^n \\ &= n^2 - n + \frac{3}{2} - \frac{1}{2}(-1)^n. \end{aligned}$$

Linear Recurrences Involving Two Sequences

- You bombard a chunk of uranium with a high-energy neutron at $t = 0$.
- Neutrons are classified as high-energy and low-energy.
- In one microsecond (μs), each neutron collides with a uranium nucleus, and before getting absorbed, generates more neutrons.
- A high-energy neutron generates two high-energy and one low-energy neutrons.
- A low-energy neutron generates one high-energy and one low-energy neutrons.
- h_n is the number of high-energy neutrons n μs after the initial bombardment.
- l_n is the number of low-energy neutrons n μs after the initial bombardment.
- We have $h_0 = 1$ and $l_0 = 0$.
- For $n \geqslant 1$, we have

$$\begin{aligned} h_n &= 2h_{n-1} + l_{n-1}, \\ l_n &= h_{n-1} + l_{n-1}. \end{aligned}$$

The Generating Functions $H(x)$ and $L(x)$

- We have

$$H(x) = \sum_{n \geqslant 0} h_n x^n = h_0 + \sum_{n \geqslant 1} (2h_{n-1} + l_{n-1}) x^n = 1 + 2xH(x) + xL(x), \text{ and}$$

$$L(x) = \sum_{n \geqslant 0} l_n x^n = l_0 + \sum_{n \geqslant 1} (h_{n-1} + l_{n-1}) x^n = xH(x) + xL(x).$$

- Solving these linear equations, we get

$$H(x) = \frac{1-x}{1-3x+x^2} = \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1}{1-\alpha x}\right) + \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1}{1-\beta x}\right), \text{ and}$$

$$L(x) = \frac{x}{1-3x+x^2} = \left(\frac{-5+3\sqrt{5}}{10}\right)\left(\frac{1}{1-\alpha x}\right) + \left(\frac{-5-3\sqrt{5}}{10}\right)\left(\frac{1}{1-\beta x}\right),$$

where $\alpha = \dfrac{3+\sqrt{5}}{2}$ and $\beta = \dfrac{3-\sqrt{5}}{2}$.

The Solution

- We have $\alpha = 2.618033\ldots > 1$ and $\beta = 0.381966\ldots < 1$. Therefore

$$h_n = \left(\frac{5-\sqrt{5}}{10}\right)\alpha^n + \left(\frac{5+\sqrt{5}}{10}\right)\beta^n \approx 0.276393 \times 2.618033^n, \text{ and}$$

$$l_n = \left(\frac{-5+3\sqrt{5}}{10}\right)\alpha^n + \left(\frac{-5-3\sqrt{5}}{10}\right)\beta^n \approx 0.170821 \times 2.618033^n.$$

- One millisecond after the initial bombardment, we have

$$\begin{aligned} h_{1000} &\approx 2.61 \times 10^{417}, \\ l_{1000} &\approx 1.61 \times 10^{417}. \end{aligned}$$

Catalan Numbers

$$\begin{aligned} C_0 &= 1, \\ C_n &= C_0 C_{n-1} + C_1 C_{n-2} + C_2 C_{n-3} + \cdots + C_{n-1} C_0 \ \text{ for all } \ n \geqslant 1. \end{aligned}$$

The generating function for the Catalan series is

$$\begin{aligned} C(x) &= C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots + C_n x^n + \cdots \\ &= 1 + C_0 C_0 x + (C_0 C_1 + C_1 C_0) x^2 + (C_0 C_2 + C_1 C_1 + C_2 C_0) x^3 + \cdots + \\ &\quad (C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0) x^n + \cdots \\ &= 1 + x(C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n + \cdots)(C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n + \cdots) \\ &= 1 + xC(x)^2. \end{aligned}$$

Solve for $C(x)$

$$xC(x)^2 - C(x) + 1 = 0.$$

$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

The choice of $+$ in the numerator gives a term $1/x$ in $C(x)$. So

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

Therefore C_n is equal to half the coefficient of x^{n+1} in the power-series expansion of the numerator $1 - \sqrt{1-4x}$.

The Closed-Form Formula for Catalan Numbers

$$
\begin{aligned}
C_n &= -(-4)^{n+1}\frac{1}{2}\left[\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)\cdots(\frac{1}{2}-n)}{(n+1)!}\right] \\
&= (-1)^{n+2}\times\frac{4^{n+1}}{2^{n+2}}\times(-1)^n\times\left[\frac{1\times3\times5\times\cdots\times(2n-1)}{(n+1)!}\right] \\
&= 2^n\times\left[\frac{1\times3\times5\times\cdots\times(2n-1)}{(n+1)!}\right] \\
&= 2^n\times\left[\frac{1\times3\times5\times\cdots\times(2n-1)\times n!}{(n+1)!n!}\right] \\
&= \frac{1\times3\times5\times\cdots\times(2n-1)\times2\times4\times6\times\cdots\times(2n)}{(n+1)!n!} \\
&= \frac{(2n)!}{(n+1)n!n!} \;=\; \frac{1}{n+1}\binom{2n}{n}.
\end{aligned}
$$