Why is it necessary to check whether an identity is both a left identity and a right identity?

In what follows, we let $*$ be an associative operation on a set S.

Theorem: If S contains a left identity L and a right identity R, then $L = R$, and this element is unique.

Proof By closure, $a = L * R \in S$. Since L is a left identity, we have $a = R$. Since R is a right identity, we have $a = L$. Therefore $a = L = R$. If L, L' are two left identities, then by the result just proved, we have $L = R$ and $L' = R$, that is, $L = L'$, that is, the left identity is unique. Analogously, the right identity is unique. ◀

From this result, it appears that checking both is not needed. But note that the theorem has a condition "if L and R both exist". Now, is it possible that this condition fails to hold?

The answer is yes. Consider the set S of all 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix},$$

where a, b are non-negative integers with $a > b$ (so $b \geqslant 0$, and $a > 0$). Take $*$ as normal matrix multiplication.

[Closure] We have the product

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ac & ad \\ 0 & 0 \end{pmatrix}.$$

Since $a > 0$ and $c > d \geqslant 0$, we have $ac > ad \geqslant 0$.

[Associativity] Matrix multiplication is associative.

[Left identity] Consider the condition

$$\begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}.$$

We then have $\alpha a = a$ and $\alpha b = b$. Since this holds for any a, b, we should have $\alpha = 1$. This immediately does not put any restriction on β. But we should have $1 = \alpha > \beta \geqslant 0$, so this forces $\beta = 0$. Therefore

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is the unique left identity in the structure.

[Right identity] Consider the condition

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}.$$

We then have $a\alpha = a$ and $a\beta = b$. Since $a > 0$, we have $\alpha = 1$. But for any fixed β, the condition $a\beta = b$ cannot be satisfied by all a, b. It follows that a right identity in this structure does not exist.

Note: If we do not impose the restriction $a > b$ in the definition, that is, if we allow a, b to be arbitrary integers (non-negative if you prefer), then any element of the form

$$\begin{pmatrix} 1 & \beta \\ 0 & 0 \end{pmatrix}$$

is a left identity. So the left identity need not be unique. A right identity still does not exist anyway, and that is precisely why the (countably infinite) army of left identities does not feel any obligation to collapse itself to equate to any right-wing specialist.

Therefore if you suspect some element as the identity, you should check that it is both a left identity and a right identity. If yes, then by the above theorem, that element will be the unique identity in the structure.

Likewise, check for both left and right inverses.