CS21201 Discrete Structures

Tutorial 7

Generating Functions

1. Find the generating function of the sequence $1, 2, 0, 3, 4, 0, 5, 6, 0, 7, 8, 0, \ldots$.

Solution Let us decompose the sequence as follows.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
		−3			−6			−9			−12	0	0	−15	...
			−1	−1		−2	−2		−3	−3		−4	−4		...
1	2	0	3	4	0	5	6	0	7	8	0	9	10	0	...

Therefore the generating function of the given sequence is

$$\begin{aligned}
& \left(1+2x+3x^2+4x^3+\cdots\right) - 3x^2\left(1+2x^3+3x^6+4x^9+\cdots\right) - (x^3+x^4)(1+2x^3+3x^6+4x^9+\cdots) \\
= \; & \frac{1}{(1-x)^2} - \frac{3x^2}{(1-x^3)^2} - \frac{x^3(1+x)}{(1-x^3)^2} \\
= \; & \frac{1+2x+x^3}{(1-x^3)^2}.
\end{aligned}$$

2. Let $A(x)$ be the generating function of the sequence $a_0, a_1, a_2, \ldots$. Express the generating function of the sequence $a_0+a_1, a_2+a_3, a_4+a_5, a_6+a_7, \ldots$ in terms of $A(\,)$.

Solution The generating function of the given sequence is

$$\begin{aligned}
& (a_0+a_1)+(a_2+a_3)x+(a_4+a_5)x^2+(a_6+a_7)x^3+\cdots \\
= \; & \left(a_0+a_2x+a_4x^2+a_6x^3+\cdots\right)+\left(a_1+a_3x+a_5x^2+a_7x^3+\cdots\right) \\
= \; & \left(\frac{A(\sqrt{x})+A(-\sqrt{x})}{2}\right)+\left(\frac{A(\sqrt{x})-A(-\sqrt{x})}{2\sqrt{x}}\right).
\end{aligned}$$

3. Let F_n, $n \geqslant 0$, denote the Fibonacci numbers. Prove that $\sum_{n\in\mathbb{N}_0} \frac{F_n}{2^n} = 2$.

Solution The generating function of the Fibonacci sequence is

$$\sum_{n\in\mathbb{N}_0} F_n x^n = \frac{x}{1-x-x^2} = \frac{x}{(1-\rho x)(1-\bar{\rho}x)} = \frac{A}{1-\rho x} + \frac{B}{1-\bar{\rho}x},$$

where $\rho = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots$ is the golden ratio, $\bar{\rho} = \frac{1-\sqrt{5}}{2} - 0.6180339887\cdots$ is the conjugate of the golden ratio, and A, B are constant real numbers. Since $|\rho/2|$ and $|\bar{\rho}/2|$ are less than 1, the power series converge for $x = \frac{1}{2}$, and we get

$$\sum_{n\in\mathbb{N}_0} \frac{F_n}{2^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}-(\frac{1}{2})^2} = 2.$$

4. Let $A(x)$ be the EGF of the sequence $a_0, a_1, a_2, \ldots$. Express the EGF of the sequence $a_1-a_0, a_2-a_1, a_3-a_2, \ldots$ in terms of $A(\,)$.

Solution The desired EGF is

$$\begin{aligned}
& (a_1-a_0)+(a_2-a_1)x+(a_3-a_2)\frac{x^2}{2!}+(a_4-a_3)\frac{x^3}{3!}+\cdots \\
= \; & \left(a_1+a_2x+a_3\frac{x^2}{2!}+a_4\frac{x^3}{3!}+\cdots\right)-\left(a_0+a_1x+a_2\frac{x^2}{2!}+a_3\frac{x^3}{3!}+\cdots\right) \\
= \; & A'(x)-A(x).
\end{aligned}$$

5. Let A be a (real-valued) discrete random variable, and $n \in \mathbb{N}_0$. The n-th moment of A (about zero) is defined as $\mu_n = \mathrm{E}[A^n]$. The exponential generating function of the sequence $\mu_0, \mu_1, \mu_2, \mu_3, \ldots$ is called the *moment generating function* $M_A(x)$ of A. Prove that $M_A(x) = \mathrm{E}[e^{xA}]$ (provided that this expectation exists).

Solution Let the space for A be a_i, $i \in I$ with $\Pr[A = a_i] = p_i$ for all $i \in I$. We have

$$\mu_n = \mathrm{E}[A^n] = \sum_{i \in I} p_i a_i^n.$$

Then, the EGF of the moments is

$$\sum_{n \geqslant 0} \mu_n \frac{x^n}{n!} = \sum_{n \geqslant 0} \left(\sum_{i \in I} p_i a_i^n \right) \frac{x^n}{n!} = \sum_{i \in I} p_i \left(\sum_{n \geqslant 0} a_i^n \frac{x^n}{n!} \right) = \sum_{i \in I} p_i e^{x a_i} = \mathrm{E}(e^{xA}).$$

6. Let a_n, $n \geqslant 0$, be the sequence satisfying

$$\begin{aligned} a_0 &= 1, \\ a_n &= 2 + 2a_0 + 2a_1 + 2a_2 + \cdots + 2a_{n-2} + a_{n-1} \text{ for } n \geqslant 1. \end{aligned}$$

Deduce that the generating function of this sequence is $\dfrac{1+x}{1-2x-x^2}$. Solve for a_n.

Solution The (ordinary) generating function of the sequence is

$$\begin{aligned} A(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \\ &= 1 + (2 + a_0)x + (2 + 2a_0 + a_1)x^2 + (2 + 2a_0 + 2a_1 + a_2)x^3 + \cdots + \\ &\quad (2 + 2a_0 + 2a_1 + 2a_2 + \cdots + 2a_{n-2} + a_{n-1})x^n + \cdots \\ &= 1 + 2x\left(1 + x + x^2 + x^3 + \cdots + x^{n-1} + \cdots\right) + 2x^2\left(a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots\right) + \\ &\quad x\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + \cdots\right) \\ &= 1 + \frac{2x}{1-x} + 2x^2\left(\frac{A(x)}{1-x}\right) + xA(x). \end{aligned}$$

It therefore follows that

$$\left(1 - \frac{2x^2}{1-x} - x\right)A(x) = 1 + \frac{2x}{1-x},$$

that is,

$$(1 - 2x - x^2)A(x) = 1 + x,$$

that is,

$$A(x) = \frac{1+x}{1-2x-x^2}.$$

Now, use the fact that $1 - 2x - x^2 = \left(1 - \left(1 + \sqrt{2}\right)x\right)\left(1 - \left(1 - \sqrt{2}\right)x\right)$.

7. The generating function $A(x)$ of a sequence $a_0, a_1, a_2, a_3, \ldots$ satisfies $A'(x) = 1 + A(x)$. Prove that $A(x) = (a_0 + 1)e^x - 1$.

Solution We have $a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = 1 + (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$. Equating the coefficients of $x^0, x^1, x^2, x^3, \ldots$ from the two sides gives $a_1 = a_0 + 1$, $2a_2 = a_1$, that is, $a_2 = (a_0 + 1)/2!$, $3a_3 = a_2$, that is, $a_3 = (a_0 + 1)/3!$, and so on.

8. A test has four sections. Section A contains many questions of 2 marks each. Section B has many questions of 5 marks each. Section C has a single question of 4 marks. Section D has a single question of 1 mark.

Assume that the questions in Sections A, B and D are of objective numerical type. You either get full marks or zero. The Section C question is essay-type, and you can get an integer mark in the range $[0,4]$. In how many ways can you get a total of n marks?

Solution The relevant generating function is

$$\begin{aligned} & (1+x^2+x^4+x^6+\cdots)(1+x^5+x^{10}+x^{15}+\cdots)(1+x+x^2+x^3+x^4)(1+x) \\ = & \left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1-x^5}{1-x}\right)(1+x) \\ = & \frac{1}{(1-x)^2} \\ = & 1+2x+3x^2+4x^3+\cdots+(n+1)x^n+\cdots. \end{aligned}$$

The desired answer is therefore $n+1$.

9. (a) For $n \in \mathbb{N}$, denote by $\sigma(n)$ the sum of all positive integral divisors of n. We also take $\sigma(0)=0$. Find the generating function of the sequence $\sigma(0), \sigma(1), \sigma(2), \ldots, \sigma(n), \ldots$.

Solution $\sum_{n \geqslant 1}(nx^n+nx^{2n}+nx^{3n}+\cdots) = \sum_{n \geqslant 1} \frac{nx^n}{1-x^n}$.

(b) If $u, v \in \mathbb{N}$ are coprime, prove that $\sigma(uv)=\sigma(u)\sigma(v)$. Hence deduce a closed-form expression for $\sigma(n)$ with n having the prime factorization $n = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$.

Solution We have $\sigma(p^e) = 1+p+p^2+\cdots+p^e = \frac{p^{e+1}-1}{p-1}$. By the multiplicative property, we therefore have

$$\sigma(n) = \prod_{i=1}^{t}\left(\frac{p_i^{e_i+1}-1}{p_i-1}\right).$$

10. Using generating functions, solve the following mutually defined recurrences.

$$\begin{aligned} a_0 &= 1, \\ a_1 &= 2, \\ b_0 &= 3, \\ a_n &= a_{n-1}+b_n \text{ for } n \geqslant 1, \\ b_n &= b_{n-1}+a_{n-2} \text{ for } n \geqslant 2. \end{aligned}$$

Solution We have

$$A(x) = a_0 + \sum_{n \geqslant 1} a_n x^n = 1 + \sum_{n \geqslant 1}(a_{n-1}+b_n)x^n = 1 + xA(x) + (B(x)-b_0) = xA(x)+B(x)-2,$$

that is,

$$(1-x)A(x)-B(x) = -2.$$

We have $b_1 = a_1 - a_0 = 1$, and so

$$\begin{aligned} B(x) &= b_0 + b_1 x + \sum_{n \geqslant 2} b_n x^n \\ &= 3 + x + \sum_{n \geqslant 2}(b_{n-1}+a_{n-2})x^n \\ &= 3 + x + x(B(x)-b_0) + x^2 A(x) \\ &= 3 - 2x + xB(x) + x^2 A(x), \end{aligned}$$

that is,

$$-x^2A(x)+(1-x)B(x)=3-2x.$$

Solve for $A(x)$ and $B(x)$ from these two linear equations.

$$\Big((1-x)^2-x^2\Big)A(x)=-2(1-x)+(3-2x),$$

that is, $A(x)=\frac{1}{1-2x}$. Consequently, $B(x)=(1-x)A(x)+2=\frac{1-x}{1-2x}+2=\frac{1/2}{1-2x}+5/2$. It follows that $a_n=2^n$ for all $n\geqslant 0$, whereas $b_n=\begin{cases}3 & \text{if } n=0,\\ 2^{n-1} & \text{if } n\geqslant 1.\end{cases}$

11. Let the two-variable sequence $a_{m,n}$ be recursively defined as follows.

$$a_{m,n}=\begin{cases}1 & \text{if } m=0 \text{ or } n=0,\\ a_{m-1,n}+a_{m,n-1} & \text{if } m\geqslant 1 \text{ and } n\geqslant 1.\end{cases}$$

Find the generating function $A(x,y)=\sum_{m,n\geqslant 0} a_{m,n}x^my^n$. From this, derive a closed-form formula for $a_{m,n}$.

Solution We have

$$\begin{aligned}A(x,y) &= \sum_{m,n\geqslant 0} a_{m,n}x^my^n\\ &= 1+\sum_{m\geqslant 1}x^m+\sum_{n\geqslant 1}y^n+\sum_{m,n\geqslant 1}(a_{m-1,n}+a_{m,n-1})x^my^n\\ &= 1+\frac{x}{1-x}+\frac{y}{1-y}+\sum_{m,n\geqslant 1}a_{m-1,n}x^my^n+\sum_{m,n\geqslant 1}a_{m,n-1}x^my^n.\end{aligned}$$

Replacing $m-1$ by m in the first sum gives

$$\sum_{m,n\geqslant 1}a_{m-1,n}x^my^n = x\sum_{\substack{m\geqslant 0\\ n\geqslant 1}}a_{m,n}x^my^n = x\left[\sum_{\substack{m\geqslant 0\\ n\geqslant 0}}a_{m,n}x^my^n\right]-x\left[\sum_{m\geqslant 0}x^m\right] = xA(x,y)-\frac{x}{1-x}.$$

Likewise, the second sum is $yA(x,y)-\frac{y}{1-y}$. Using these expressions gives

$$A(x,y)=\frac{1}{1-x-y}=1+(x+y)+(x+y)^2+(x+y)^3+\cdots+(x+y)^{m+n}+\cdots.$$

This gives $a_{m,n}=\binom{m+n}{m}=\binom{m+n}{n}$.

Additional Exercises

12. Find the generating functions of the following sequences.

(a) $1,0,0,1,0,0,1,0,0,\ldots$
(b) $1,1,0,1,1,0,1,1,0,\ldots$
(c) $1,3,5,7,9,11,13,\ldots$
(d) $2,4,8,14,22,32,44,\ldots$
(e) $1,0,0,2,0,0,3,0,0,4,0,0,\ldots$
(f) $1,2,0,3,4,0,5,6,0,7,8,0,\ldots$
(g) $1/1,1/2,1/3,1/4,1/5,\ldots$
(h) $H_0,H_1,H_2,H_3,\ldots$ (where H_n is the n-th harmonic number)

13. Let $A(x)$ be the generating function for the sequence $a_0,a_1,a_2,\ldots$. Express the generating functions of the following sequences in terms of $A(x)$.

(a) $a_0, 2a_1, 3a_2, 4a_3, 5a_4, \ldots$
(b) $a_0, a_1 - a_0, a_2 - a_1, a_3 - a_2, \ldots$
(c) $a_0, a_0 - a_1, a_0 - a_1 + a_2, a_0 - a_1 + a_2 - a_3, \ldots$
(d) $a_0, a_1, a_2 - a_0, a_3 - a_1, a_4 - a_2 + a_0, a_5 - a_3 + a_1, a_6 - a_4 + a_2 - a_0, a_7 - a_5 + a_3 - a_1, \ldots$
(e) $a_0 + a_1, a_1 + a_2, a_2 + a_3, a_3 + a_4, \ldots$
(f) $a_0 + a_1, a_2 + a_3, a_4 + a_5, a_6 + a_7, \ldots$

14. Let $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$ be the sequence generated by $\sum_{r \in \mathbb{N}} \frac{x^r}{1 - x^r}$. Denote by p_n the parity of a_n, that is, $p_n = a_n \pmod 2$, that is, $p_n = \begin{cases} 0 & \text{if } a_n \text{ is even,} \\ 1 & \text{if } a_n \text{ is odd.} \end{cases}$ Determine all $n \in \mathbb{N}$, for which $p_n = 1$. Justify.

15. Let $a_n = \sum_{i=n}^{\infty} \frac{2^i}{i!}$ for all integers $n \geqslant 0$.

(a) Find a closed-form expression for the (ordinary) generating function $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$ of the sequence $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$.

(b) Use the expression for $A(x)$ in Part (a) to prove that $\sum_{n=0}^{\infty} a_n = 3e^2$.

16. Let $m \geqslant 1$ be an integer constant. Let $b_n^{(m)}$ denote the number of ordered partitions (that is, compositions) of the integer $n \geqslant 0$ such that no summand is larger than m.

(a) Prove that the (ordinary) generating function of $b_n^{(m)}$ is

$$B^{(m)}(x) = \frac{1 - x}{1 - 2x + x^{m+1}}.$$

(b) From the formula of Part (a), deduce that $b_n^{(2)} = F_{n+1}$, where $F_0, F_1, F_2, \ldots$ is the sequence of Fibonacci numbers.

17. Let l_n be the number of lines printed by the call $f(n)$ for some integer $n \geqslant 0$.

```
void f ( int n )
{
    int i, j;
    printf("Hi\n");
    if (n == 0) return;
    for (i=0; i<=n-1; ++i)
        for (j=0; j<=i; ++j)
            f(j);
}
```

(a) Let $L(x) = l_0 + l_1 x + l_2 x^2 + \cdots + l_n x^n + \cdots$ be the generating function of the sequence $l_0, l_1, l_2, \ldots$. Prove that $L(x) = \frac{1 - x}{1 - 3x + x^2}$.

(b) Derive an explicit formula for l_n from the generating function $L(x)$.

18. Find the number of solutions of $x_1 + x_2 + x_3 + x_4 = n$ with integer-valued variables satisfying $x_1 \geqslant -1$, $x_2 \geqslant -2$, $x_3 \geqslant 3$, and $x_4 \geqslant 4$.

19. How many bit strings of length n are there in which 1's always occur in contiguous pairs? You should consider strings of the form 0011011110, but not of the form 0110111110, because the last 1 is not paired.

20. Use generating functions to prove that every positive integer has a unique binary representation (without leading zero bits).

21. Let $A(x)$ be the generating function of the sequence $a_0, a_1, a_2, a_3, \ldots$ of real numbers. Prove that $1/A(x)$ is the generating function of a sequence if and only if $a_0 \neq 0$.

22. **(a)** Find the probability generating function of the binomial distribution $\Pr[B_{n,p} = r] = \binom{n}{r} p^r (1-p)^{n-r}$ for $r = 0, 1, 2, \ldots, n$. Hence deduce the expectation $\mathrm{E}[B_{n,p}]$.

(b) Find the probability generating function of the uniform distribution $\Pr[U_{a,b} = r] = \dfrac{1}{b-a+1}$ for $r = a, a+1, a+2, \ldots, b$ (where $a, b \in \mathbb{Z}$ with $a \leqslant b$). Hence deduce the expectation $\mathrm{E}[U_{a,b}]$.

23. Let $A(x)$ be the exponential generating function of a sequence $a_0, a_1, a_2, \ldots$. Express the exponential generating functions of the following sequences in terms of $A(x)$.

(a) $a_0, 2a_1, 3a_2, 4a_3, 5a_4, \ldots$
(b) $0, a_0, a_1, a_2, a_3, a_4, \ldots$
(c) $a_1, a_2, a_3, a_4, a_5, \ldots$
(d) $a_0, a_1 - a_0, a_2 - a_1, a_3 - a_2, \ldots$

24. Let $A(x)$ and $B(x)$ be the exponential generating functions of the two sequences $a_0, a_1, a_2, a_3, \ldots$ and $b_0, b_1, b_2, b_3, \ldots$. Of what sequence is $A(x)B(x)$ the EGF?

25. Prove the following identities involving the Stirling numbers $S(n,k)$ of the second kind. Take $S(0,0) = 1$, and $S(n,k) = 0$ for $n < k$.

(a) $\displaystyle\sum_{n \in \mathbb{N}_0} S(n,k) x^n = \prod_{r=1}^{k} \frac{x}{1-rx}$.

(b) $\displaystyle\sum_{n \in \mathbb{N}_0} S(n,k) \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}$.

26. Deduce that the exponential generating function of the Bell numbers B_n, $n \in \mathbb{N}_0$, is $e^{e^x - 1}$.

27. Find the moment generating functions of the following random variables.

(a) Binomial distribution: $\Pr[B_{n,p} = r] = \binom{n}{r} p^r (1-p)^r$ for $r = 0, 1, 2, \ldots, n$.
(b) Geometric distribution: $\Pr[G_p = r] = (1-p)^{r-1} p$ for $r \in \mathbb{N}$.
(c) The discrete uniform distribution: $\Pr[U_{a,b} = r] = \dfrac{1}{b-a+1}$ for $r = a, a+1, a+2, \ldots, b$.
(d) The continuous uniform distribution: $\Pr[C_{a,b} = r] = \dfrac{1}{b-a}$ for $r \in [a,b]$.
(e) Poisson distribution: $\Pr[P_\lambda = r] = e^{-\lambda} \dfrac{\lambda^r}{r!}$ for $r \in \mathbb{N}_0$.

28. Generating functions with multiple variables are sometimes used. Suppose that r elements are chosen from $\{1, 2, 3, \ldots, n\}$ such that the sum of the chosen elements is s. We want to count how many such collections are possible. Argue that this count is the coefficient of $x^r y^s$ of $(1+xy)(1+xy^2)(1+xy^3)\cdots(1+xy^n)$. Find the two-variable generating function of these counts if the r elements are chosen from $\{1, 2, 3, \ldots, n\}$ with repetitions allowed.

29. Let a_n, $n \geqslant 0$, be the sequence satisfying

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 1, \\ a_n &= a_{n-1} + \sum_{k=1}^{n-2} a_k a_{n-1-k} \quad \text{for } n \geqslant 2. \end{aligned}$$

Prove that the generating function for this sequence is $\dfrac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x}$. Solve for a_n.

30. Solve the following recurrence relation using generating functions: $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, $a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3} + 1$ for $n \geqslant 3$.