CS21201 Discrete Structures

Tutorial 2

Propositional Logic

1. Which of the following sentences are valid (tautology), unsatisfiable, or neither.

(a) $\texttt{Smoke} \to \texttt{Smoke}$

Solution $\texttt{Smoke} \to \texttt{Smoke} \equiv \neg\texttt{Smoke} \vee \texttt{Smoke} \equiv \top$ (valid)

(b) $\texttt{Smoke} \to \texttt{Fire}$

Solution Consider $\texttt{Smoke} = \top$ and $\texttt{Fire} = \bot$ (invalid)
Consider $\texttt{Smoke} = \top$ and $\texttt{Fire} = \top$ (satisfiable)

(c) $\texttt{Smoke} \vee \texttt{Fire} \vee \neg\,\texttt{Fire}$

Solution $\texttt{Smoke} \vee \texttt{Fire} \vee \neg\texttt{Fire} \equiv \texttt{Smoke} \vee \top \equiv \top$ (valid)

(d) $(\texttt{Smoke} \wedge \neg\texttt{Fire}) \to \texttt{Fire}$

Solution Consider $\texttt{Smoke} = \top$ and $\texttt{Fire} = \bot$ (invalid)
Consider $\texttt{Smoke} = \top$ and $\texttt{Fire} = \top$ (satisfiable)

(e) $(\texttt{Smoke} \to \texttt{Fire}) \to (\neg\texttt{Smoke} \to \neg\texttt{Fire})$

Solution Consider $\texttt{Smoke} = \bot$ and $\texttt{Fire} = \top$ (invalid)
Consider $\texttt{Smoke} = \bot$ and $\texttt{Fire} = \bot$ (satisfiable)

(f) $(\texttt{Smoke} \to \texttt{Fire}) \to ((\texttt{Smoke} \wedge \texttt{Heat}) \to \texttt{Fire})$

Solution
$$(\texttt{Smoke} \to \texttt{Fire}) \to ((\texttt{Smoke} \wedge \texttt{Heat}) \to \texttt{Fire})$$
$$\equiv (\texttt{Smoke} \wedge \neg\texttt{Fire}) \vee (\neg\texttt{Smoke} \vee \neg\texttt{Heat} \vee \texttt{Fire})$$
$$\equiv (\texttt{Smoke} \vee \neg\texttt{Smoke} \vee \neg\texttt{Heat} \vee \texttt{Fire}) \wedge (\neg\texttt{Fire} \vee \neg\texttt{Smoke} \vee \neg\texttt{Heat} \vee \texttt{Fire})$$
$$\equiv (\top \vee \neg\texttt{Heat} \vee \texttt{Fire}) \wedge (\top \vee \neg\texttt{Smoke} \vee \neg\texttt{Heat}) \equiv \top$$ (valid)

2. Prove the following two logical deductions.

$$\begin{array}{c} (\neg p \vee q) \to r \\ r \to (s \vee t) \\ \neg(s \vee u) \\ t \to u \\ q \leftrightarrow v \\ (v \wedge \neg w) \vee (\neg v \wedge w) \to \neg p \\ \hline \therefore\ \neg w \end{array} \qquad\qquad \begin{array}{c} t \to q \\ \neg r \to \neg s \\ p \to u \\ \neg t \to \neg r \\ u \to s \\ \hline \therefore\ p \to q \end{array}$$

Solution The first deduction goes as follows.

$$\begin{array}{c} \neg(s \vee u) \\ \hline \therefore\ \neg s\ ,\ \neg u \end{array} \qquad \begin{array}{c} \neg u \\ t \to u \\ \hline \therefore\ \neg t \end{array} \qquad \begin{array}{c} r \to (s \vee t) \\ \neg s\ ,\ \neg t \\ \hline \therefore\ \neg r \end{array} \qquad \begin{array}{c} (\neg p \vee q) \to r \\ \neg r \\ \hline \therefore\ p \wedge \neg q \end{array} \qquad \begin{array}{c} q \leftrightarrow v \\ \neg q \\ \hline \therefore\ \neg v \end{array}$$

$$\begin{array}{c} (v \wedge \neg w) \vee (\neg v \wedge w) \to \neg p \\ p \\ \hline \therefore\ (\neg v \vee w) \wedge (v \vee \neg w) \end{array} \qquad\qquad \begin{array}{c} (\neg v \vee w) \wedge (v \vee \neg w) \\ \neg v \\ \hline \therefore\ \neg w \end{array}$$

The second deduction goes as follows.

$$\frac{\begin{array}{c} p \to u \\ u \to s \end{array}}{\therefore\ p \to s} \qquad \frac{\begin{array}{c} p \to s \\ \neg r \to \neg s \equiv s \to r \end{array}}{\therefore\ p \to r} \qquad \frac{\begin{array}{c} p \to r \\ \neg t \to \neg r \equiv r \to t \end{array}}{\therefore\ p \to t} \qquad \frac{\begin{array}{c} p \to t \\ t \to q \end{array}}{\therefore\ p \to q}$$

3. Encode and reason about the following.

If a scarcity of commodities develops, then the prices rise. If there is a change of government, then fiscal controls will not be continued. If the threat of inflation persists, then fiscal controls will be continued. If there is over-production, then prices do not rise. It has been found that there is over-production and there is a change of government. Therefore, neither the scarcity of commodities has developed, nor there is a threat of inflation.

Solution The propositions are as follows:

- sc : a scarcity of commodities develops
- pr : the prices rise
- cg : there is a change of government
- fc : fiscal controls will be continued
- ti : the threat of inflation persists
- op : there is over-production

The propositional logic encoding of the statements and logical deduction are as follows:

$$\frac{\begin{array}{lcl} F_1 & : & \mathrm{sc} \to \mathrm{pr} \\ F_2 & : & \mathrm{cg} \to \neg\mathrm{fc} \\ F_3 & : & \mathrm{ti} \to \mathrm{fc} \\ F_4 & : & \mathrm{op} \to \neg\mathrm{pr} \\ F_5 & : & \mathrm{op} \wedge \mathrm{cg} \end{array}}{G\ :\ \neg\mathrm{sc} \wedge \neg\mathrm{ti}} \qquad \frac{\begin{array}{c} \mathrm{cg} \to \neg\mathrm{fc} \\ \mathrm{op} \wedge \mathrm{cg} \end{array}}{\therefore\ \neg\mathrm{fc}} \qquad \frac{\begin{array}{c} \mathrm{op} \to \neg\mathrm{pr} \\ \mathrm{op} \wedge \mathrm{cg} \end{array}}{\therefore\ \neg\mathrm{pr}} \qquad \frac{\begin{array}{c} \mathrm{sc} \to \mathrm{pr} \\ \neg\mathrm{pr} \end{array}}{\therefore\ \neg\mathrm{sc}} \qquad \frac{\begin{array}{c} \mathrm{ti} \to \mathrm{fc} \\ \neg\mathrm{fc} \end{array}}{\therefore\ \neg\mathrm{ti}}$$

Additional Exercises

4. Formalize these statements, and determine (with truth tables or otherwise) whether they are consistent (that is, if there are some assumptions on the atomic propositions that make it true).

The system is in a multiuser state iff it is operating normally. If the system is operating normally, the kernel is functioning. Either the kernel is not functioning or the system is in interrupt mode. If the system is not in multiuser state, then it is in interrupt mode. The system is not in interrupt mode.

5. Encode and deduce using logical inferencing techniques.

While walking in a labyrinth, you find yourself in front of three possible roads. The road on your left is paved with gold, the road in front of you is paved with marble, while the road on your right is made of small stones. Each road is protected by a guard. You talk to the guards, and they tell you this:

- *The guard of the gold road: "This road will bring you straight to the center. Moreover, if the stones take you to the center, then also the marble takes you to the center."*
- *The guard of the marble road: "Neither the gold nor the stones will take you to the center."*
- *The guard of the stone road: "Follow the gold, and you will reach the center. Follow the marble, and you will be lost."*

You know that all the guards are liars. Your goal is to choose the correct road that will lead you to the center of the labyrinth.

6. Prove that the negation of the statement $\exists x \forall y \big[P(x,y) \to \neg Q(y)\big]$ is $\forall x \exists y \big[P(x,y) \land Q(y)\big]$.

Solution $\neg \exists x \forall y \big[P(x,y) \to \neg Q(y)\big] \equiv \forall x \exists y \neg \big[\neg P(x,y) \lor \neg Q(y)\big] \equiv \forall x \exists y \big[P(x,y) \land Q(y)\big]$

7. Prove that $\forall x \big[P(x) \to (Q(x) \leftrightarrow R(x))\big]$ is equivalent to

$$\Big[\forall x \big[(P(x) \land Q(x)) \to R(x)\big]\Big] \land \Big[\forall x \big[(P(x) \land R(x)) \to Q(x)\big]\Big].$$

Solution $\Big[\forall x \big[(P(x) \land Q(x)) \to R(x)\big]\Big] \land \Big[\forall x \big[(P(x) \land R(x)) \to Q(x)\big]\Big]$

$$\equiv \forall x \Big[\big[(P(x) \land Q(x)) \to R(x)\big] \land \big[(P(x) \land R(x)) \to Q(x)\big]\Big]$$

$$\equiv \forall x \Big[\big[\neg P(x) \lor \neg Q(x) \lor R(x)\big] \land \big[\neg P(x) \lor \neg R(x) \lor Q(x)\big]\Big]$$

$$\equiv \forall x \Big[\neg P(x) \lor \big[(\neg Q(x) \lor R(x)) \land (\neg R(x) \lor Q(x))\big]\Big]$$

$$\equiv \forall x \Big[P(x) \to \big[(Q(x) \to R(x)) \land (R(x) \to Q(x))\big]\Big] \equiv \forall x \big[P(x) \to (Q(x) \leftrightarrow R(x))\big]$$

8. Formalize the following sentences in first-order logic using only the four predicates given below.

(i) $\text{inside}(x,y)$: x is inside of y; (ii) $\text{free}(x)$: x is free;
(iii) $\text{love}(x,y)$: x loves y; (iv) $\text{diff}(x,y)$: x differs from y.

(a) Something is inside of everything.

Solution $\exists x \forall y(\text{inside}(x,y))$

(b) Everything that is free has nothing inside it.

Solution $\forall x \big[\text{free}(x) \to \neg \exists y(\text{inside}(y,x))\big]$

(c) There is something that is inside and is not free.

Solution $\exists x \big[\forall y(\text{inside}(x,y)) \land \neg \text{free}(x)\big]$

(d) There is at least one person who loves Mary.

Solution $\exists x \big[\text{loves}(x, \text{Mary})\big]$

(e) There is exactly one person who loves Mary.

Solution $\exists x \big[\text{loves}(x, \text{Mary}) \land \forall y(\text{diff}(x,y) \to \neg \text{loves}(y, \text{Mary}))\big]$

(f) There is at most one person who loves Mary.

Solution $\forall x \big[\neg \text{loves}(x, \text{Mary})\big] \lor \exists x \big[\text{loves}(x, \text{Mary}) \land \forall y(\text{diff}(x,y) \to \neg \text{loves}(y, \text{Mary}))\big]$

Additional Parts

(g) There are exactly two persons who love Mary.
(h) Only Mary loves Bob.
(i) If Bob loves everyone that Mary loves, and Bob loves David, then Mary does not love David.

9. Encode the following logical statements using predicate logic (formulate suitable predicate and function symbols as required), and conclude on the validity of the last statement.

(a) *Every athlete is strong. Everyone who is strong and intelligent will succeed in career. Hima is an athlete. Hima is intelligent. Therefore, Hima will succeed in career.*

Solution The predicate logic encoding of the statements are as follows:

$$
\begin{aligned}
F_1 &: \forall x\big[\text{athlete}(x) \rightarrow \text{strong}(x)\big] \\
F_2 &: \forall x\big[(\text{strong}(x) \wedge \text{intelligent}(x)) \rightarrow \text{succeed}(x)\big] \\
F_3 &: \text{athlete}(\text{Hima}) \\
F_4 &: \text{intelligent}(\text{Hima}) \\
G &: \text{succeed}(\text{Hima})
\end{aligned}
$$

The logical deduction for $(F_1 \wedge F_2 \wedge F_3 \wedge F_4) \rightarrow G$ is as follows.

$$
\begin{array}{c}
\text{athlete}(x) \rightarrow \text{strong}(x) \\
\text{athlete}(\text{Hima}) \\
\hline
\therefore\ \text{strong}(\text{Hima})
\end{array}
\qquad
\begin{array}{c}
(\text{strong}(x) \wedge \text{intelligent}(x)) \rightarrow \text{succeed}(x) \\
\text{strong}(\text{Hima}) \quad , \quad \text{intelligent}(\text{Hima}) \\
\hline
\therefore\ \text{succeed}(\text{Hima})
\end{array}
$$

(b) *No man who is a candidate will be defeated if he is a good campaigner. Any man who runs for office is a candidate. Any candidate who is not defeated will be elected. Every man who is elected is a good campaigner. Therefore, Any man who runs for office will be elected if and only if he is a good campaigner.*

Solution The predicate logic encoding of the statements are as follows.

$$
\begin{aligned}
F_1 &: \forall x\big[(\text{cand}(x) \wedge \text{camp}(x)) \rightarrow \neg\text{def}(x)\big] \\
F_2 &: \forall x\big[\text{off}(x) \rightarrow \text{cand}(x)\big] \\
F_3 &: \forall x\big[(\text{cand}(x) \wedge \neg\text{def}(x)) \rightarrow \text{elec}(x)\big] \\
F_4 &: \forall x\big[\text{elec}(x) \rightarrow \text{camp}(x)\big] \\
G &: \forall x\big[\text{off}(x) \rightarrow (\text{elec}(x) \leftrightarrow \text{camp}(x))\big] \equiv G_1 \wedge G_2, \quad \text{where} \\
G_1 &: \forall x\big[(\text{off}(x) \wedge \text{elec}(x)) \rightarrow \text{camp}(x)\big] \quad \text{and} \\
G_2 &: \forall x\big[(\text{off}(x) \wedge \text{camp}(x)) \rightarrow \text{elec}(x)\big]
\end{aligned}
$$

We need to prove the logical deduction $(F_1 \wedge F_2 \wedge F_3 \wedge F_4) \rightarrow (G_1 \wedge G_2)$.

To prove $G_1 : \forall x\big[(\text{off}(x) \wedge \text{elec}(x)) \rightarrow \text{camp}(x)\big]$, proceed as follows.

$$
\begin{array}{c}
\text{off}(x) \wedge \text{elec}(x) \\
\hline
\therefore\ \text{elec}(x)
\end{array}
\qquad
\begin{array}{c}
\text{elec}(x) \rightarrow \text{camp}(x) \\
\text{elec}(x) \\
\hline
\therefore\ \text{camp}(x)
\end{array}
$$

To prove $G_2 : \forall x\big[(\text{off}(x) \wedge \text{camp}(x)) \rightarrow \text{elec}(x)\big]$, proceed as follows.

$$
\begin{array}{c}
\text{off}(x) \wedge \text{camp}(x) \\
\hline
\therefore\ \text{off}(x) \quad , \quad \text{camp}(x)
\end{array}
\qquad
\begin{array}{c}
\text{off}(x) \rightarrow \text{cand}(x) \\
\text{off}(x) \\
\hline
\therefore\ \text{cand}(x)
\end{array}
$$

$$
\begin{array}{c}
(\text{cand}(x) \wedge \text{camp}(x)) \rightarrow \neg\text{def}(x) \\
\text{cand}(x) \quad , \quad \text{camp}(x) \\
\hline
\therefore\ \neg\text{def}(x)
\end{array}
\qquad
\begin{array}{c}
(\text{cand}(x) \wedge \neg\text{def}(x)) \rightarrow \text{elec}(x) \\
\text{cand}(x) \quad , \quad \neg\text{def}(x) \\
\hline
\therefore\ \text{elec}(x)
\end{array}
$$

Additional Parts

(c) *Jack owns a dog. Every dog owner is an animal lover. No animal lover kills an animal. Either Jack or Curiosity killed Tuna, which is a cat. Did Curiosity kill the cat?*

(d) *All members are both officers and gentlemen. All officers are fighters. Only a pacifist is a gentleman or not a fighter. No pacifist is a gentleman if he is a fighter. Some members are fighters iff they are officers. Therefore, not all members are fighters.*

(e) *Everything that dies decomposes. Everything that decomposes divides into parts. Only material things are divisible into parts. No human souls are material. Therefore, no human souls die.*

10. Translate the following into idiomatic English.

(a) $\forall x \Big[\big[H(x) \wedge \forall y \neg M(x,y) \big] \to U(x) \Big]$, where $H(x)$ means x is a man, $M(x,y)$ means x is married to y, $U(x)$ means x is unhappy, and x and y range over people.

Solution All unmarried men are unhappy.

(b) $\exists z \exists x \big[P(z,x) \wedge \forall y (S(z,y) \wedge W(y)) \big]$, where $P(z,x)$ means z is a parent of x, $S(z,y)$ means z and y are siblings, $W(y)$ means y is a woman, and x, y, and z range over people.

Solution Some parents have all their siblings as women.

Additional Exercise

11. For each part, if the formula is valid, show a proof in natural deduction. If not, provide a counter-example.

(a) $\forall x \big[P(x) \to \exists y P(y) \big]$

(b) $\exists x \big[P(x) \to \forall y P(y) \big]$

(c) $\neg\neg \forall x \big[P(x) \big] \to \forall x \big[\neg\neg P(x) \big]$

(d) $\Big[\forall x \big[P(x) \to \exists y Q(x,y) \big] \Big] \to \Big[\exists x P(x) \to \exists y Q(x,y) \Big]$

(e) $\Big[\exists x Q(x) \wedge \big[\forall x (P(x) \to \neg Q(x)) \big] \Big] \to \exists x \big[\neg P(x) \big]$