

**Elementary Counting Techniques**

1. You are given  $r$  red balls,  $g$  green balls, and  $b$  blue balls. Assume that  $r, g, b$  are positive integers. Your task is to arrange the balls on a line subject to the following conditions. Find the count of all possible arrangements in each case.

(a) All blue balls appear together.

*Solution* Treat the  $b$  balls as a single object. The count is therefore  $\frac{(r+g+1)!}{r!g!}$ .

(b) The arrangement must start with a green ball and end with a non-green ball.

*Solution* Put a green ball at the beginning and a red/blue ball at the end. The count is therefore  $\frac{(r-1+g-1+b)!}{(r-1)!(g-1)!b!} + \frac{(r+g-1+b-1)!}{r!(g-1)!(b-1)!} = \frac{(r+g+b-2)!}{(r-1)!(g-1)!(b-1)!} \left( \frac{1}{r} + \frac{1}{b} \right)$ .

(c) No two red balls appear together.

*Solution* The  $g$  green balls and the  $b$  blue balls can be arranged in  $\frac{(g+b)!}{g!b!}$  ways. Insert the  $r$  red balls in between or at the two ends without repetition. The count is therefore  $\frac{(g+b)!}{g!b!} \binom{g+b+1}{r}$ .

(d) No blue ball can appear after any red ball.

*Solution* Put all the  $b$  blue balls followed by all the  $r$  red balls. Insert the  $g$  green balls in between or at the two ends with repetitions allowed. The count is therefore  $\binom{b+r+1+g-1}{g} = \binom{r+g+b}{g}$ .

2. How many sorted arrays of size  $n$  are there if each element of the array is an integer in the range  $1, 2, 3, \dots, r$ ?

*Solution* Let there be  $x_i$  occurrences of  $i \in \{1, 2, 3, \dots, r\}$ . We have  $x_1 + x_2 + x_3 + \dots + x_r = n$  for  $x_i \geq 0$ . The answer is therefore  $\binom{n+r-1}{n} = \binom{n+r-1}{r-1}$ .

3. How many binary strings of length  $n$  are there with exactly  $k$  occurrences of the pattern 01? Assume that  $n \geq 2k$ .

*Solution* All such strings are of the form

$$\begin{array}{cccccccccccccccc} 1^* & 0^* & 01 & 1^* & 0^* & 01 & 1^* & 0^* & 01 & 1^* & 0^* & \dots & 1^* & 0^* & 01 & 1^* & 0^* \\ x_1 & x_2 & 2 & x_3 & x_4 & 2 & x_5 & x_6 & 2 & x_7 & x_8 & \dots & x_{2k-1} & x_{2k} & 2 & x_{2k+1} & x_{2k+2} \end{array}$$

That is, we want to count all non-negative integer solutions of

$$x_1 + x_2 + 2 + x_3 + x_4 + 2 + x_5 + x_6 + 2 + x_7 + x_8 + \dots + x_{2k-1} + x_{2k} + 2 + x_{2k+1} + x_{2k+2} = n,$$

that is, of

$$x_1 + x_2 + \dots + x_{2k+2} = n - 2k.$$

The answer is therefore  $\binom{(n-2k) + (2k+2) - 1}{2k+2-1} = \binom{n+1}{2k+1}$ .

4. Prove the following identity for any positive integer  $n$ .

$$2^n = \binom{n+1}{1} + \binom{n+1}{3} + \binom{n+1}{5} + \dots + \begin{cases} \binom{n+1}{n+1} & \text{if } n \text{ is even,} \\ \binom{n+1}{n} & \text{if } n \text{ is odd.} \end{cases}$$

*Solution* Vary  $k$  in the range  $0, 1, 2, \dots, \lfloor n/2 \rfloor$  in the last exercise. (This exercise can be solved using other methods, like the binomial theorem.)

5. Consider paths from  $(0,0)$  to  $(n,n)$  in an  $n \times n$  grid, that never cross the diagonal. Impose an additional constraint that these paths are not allowed to touch the main diagonal except only at the beginning and at the end. How many such constrained paths are there?

*Solution*  $C(n-1)$ .

6. Suppose that  $m > n$ . How many paths from  $(0,0)$  to  $(m,n)$  with  $R$  and  $U$  movements are possible such that at no point of time, there are more  $U$  moves than  $R$  moves?

*Solution* Proceed as in the derivation of Catalan numbers. The answer is  $\binom{m+n}{n} - \binom{m+n}{n-1} = \frac{m+n-1}{m+1} \binom{m+n}{n}$ .

### Additional Exercises

7. How many subsets of size  $k$  of  $\{1, 2, 3, \dots, n\}$  are there, that contain more odd numbers than even numbers?
8. How many subsets of  $\{1, 2, 3, \dots, n\}$  contain exactly  $k$  Fibonacci numbers?
9. In how many ways you can express 100 as a sum

$$a_1 + a_2 + a_3 + \dots + a_r = 100$$

for some  $r$  with each  $a_i \in \{1, 2, 3\}$  and with  $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_r$ .

10. How many paths from  $(0,0)$  to  $(n,n)$  are possible with only  $R$  and  $U$  movements such that the paths never go above the line  $y = x + 1$ ?
11. How many paths from  $(0,0)$  to  $(n,n)$  are possible with only  $R$  and  $U$  movements such that the paths lie entirely within the two lines  $y = x - 1$  and  $y = x + 1$ ? Touching these two lines is allowed.