CS21201 Discrete Structures, Autumn 2021–2022

Third Test

Date: Nov 16, 2021 | Time: 10:15am–11:30am | Maximum marks: 40

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1. Consider the following three sets.

$$\begin{aligned} S &= \text{The set of all infinite bit sequences,} \\ A &= \text{The set of all infinite bit sequences containing two consecutive 0's (at least once),} \\ B &= \text{The set of all infinite bit sequences not containing two consecutive 0's.} \end{aligned}$$

In the class, we have seen that S is uncountable. This exercise deals with the countability/uncountability of A and B.

(a) Propose an *injective* map $f : S \to A$, and argue about the countability/uncountability of A. **(5)**

Solution Take an element (an infinite bit sequence) $\alpha \in S$. If α does not contain 0, that is, if $\alpha = 111\ldots1\ldots$, take $f(\alpha) = 00111\ldots1\ldots$. Otherwise, let $\hat{\alpha}$ be the string obtained by duplicating the *first* 0 in α, and define $f(\alpha) = 1\hat{\alpha}$. For example, $f(010101\ldots) = 10010101\ldots$, $f(0111\ldots1\ldots) = 100111\ldots1\ldots$, and $f(000\ldots0\ldots) = 10000\ldots0\ldots$.

The injective map implies $|S| \leqslant |A|$. But S is already uncountable, so A is uncountable too.

(b) Prove whether B is countable or uncountable. **(5)**

Solution *B is uncountable.* Let T be the set of all infinite sequences over the alphabet $\{1,2\}$. Define a map $g : T \to B$ as follows. Take any $\beta \in T$, and replace every occurrence of 2 by 01 to get the string $\hat{\beta}$, and define $g(\beta) = \hat{\beta}$. Since we replace all occurrences of 2 by 01, $\hat{\beta}$ does not contain any 2, so $\hat{\beta} \in S$. Moreover, the construction does not introduce two consecutive 0's in $\hat{\beta}$, so $\hat{\beta} \in B$ too, that is, g is well-defined. Finally, it is easy to see that g is injective (indeed g is a bijection). It follows that $|T| \leqslant |B|$, and like A, the set B is uncountable too.

2. Consider the sequence $a_0, a_1, a_2, \ldots$ defined recursively as follows.

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 1, \\ a_2 &= 2, \\ a_n &= 2a_{n-2} + a_{n-3} + 2 \text{ for all } n \geqslant 3. \end{aligned}$$

(a) Derive a closed-form expression for the (ordinary) generating function $A(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$ of the sequence. **(5)**

Solution We have

$$\begin{aligned} A(x) &= a_0 + a_1x + a_2x^2 + \sum_{n\geqslant 3} a_nx^n \\ &= x + 2x^2 + \sum_{n\geqslant 3} (2a_{n-2} + a_{n-3} + 2)x^n \\ &= x + 2x^2 + 2x^2 \sum_{n\geqslant 3} a_{n-2}x^{n-2} + x^3 \sum_{n\geqslant 3} a_{n-3}x^{n-3} + 2x^3 \sum_{n\geqslant 3} x^{n-3} \\ &= x + 2x^2 + 2x^2(A(x) - 0) + x^3A(x) + \frac{2x^3}{1-x} \\ &= (2x^2 + x^3)A(x) + \frac{x+x^2}{1-x}. \end{aligned}$$

$$A(x) = \frac{x+x^2}{(1-x)(1-2x^2-x^3)} = \frac{x(1+x)}{(1-x)(1+x)(1-x-x^2)} = \frac{x}{(1-x)(1-x-x^2)}.$$

(b) From the closed-form expression of $A(x)$ derived in Part (a), establish that $a_n = F_{n+2} - 1$ for all $n \geqslant 0$, where $F_0, F_1, F_2, \ldots$ is the Fibonacci sequence. Use no other method. **(5)**

Solution We can write

$$\frac{x}{(1-x)(1-x-x^2)} = \frac{A}{1-x} + \frac{Bx+c}{1-x-x^2}.$$

Solving gives $A=-1$ and $B=C=1$, that is,

$$A(x) = \frac{1+x}{1-x-x^2} - \frac{1}{1-x}.$$

The OGF of the Fibonacci sequence is $\frac{x}{1-x-x^2}$, that is, $\frac{x}{1-x-x^2}$ generates $F_0, F_1, F_2, \ldots, F_n, \ldots$. This implies that $\frac{1}{1-x-x^2}$ generates $F_1, F_2, F_3, \ldots, F_{n+1}, \ldots$. Finally, $\frac{1}{1-x}$ generates $1, 1, 1, \ldots$. Therefore, we have $a_n = F_n + F_{n+1} - 1 = F_{n+2} - 1$ for all $n \geqslant 0$.

3. Solve the following recurrence, and obtain the closed-form expression for a_n.

$$a_n = 8a_{n-2} - 16a_{n-4} + 2^n \quad (\text{for } n \geqslant 4) \quad \text{with} \quad a_0 = 1,\ a_1 = \frac{17}{4},\ a_2 = 30,\ a_3 = 41.$$

Note: Use of generating functions is **not** allowed in this exercise. **(10)**

Solution The characteristics equation for the homogeneous part of the given recurrence is:

$$r^4 - 8r^2 + 16 = 0 \quad \Rightarrow \quad (r-2)^2(r+2)^2 = 0$$

which derives the four roots as, 2, 2, -2, and -2. Hence, the general form of the homogeneous solution is:

$$a_n^{(h)} = (A+Bn)2^n + (C+Dn)(-2)^n.$$

From the given recurrence, we also get the general form of the particular solution is:

$$a_n^{(p)} = En^2 2^n.$$

Solving for the particular solution constant from the following:

$$En^2 2^n = 8E(n-2)^2 2^{n-2} - 16E(n-4)^2 2^{n-4} + 2^n, \qquad \text{we get} \quad E = \frac{1}{8}.$$

Therefore, the final generic solution form is: $\left(a_n = a_n^{(h)} + a_n^{(p)}\right)$

$$a_n = (A+Bn)2^n + (C+Dn)(-2)^n + \frac{1}{8}n^2 2^n.$$

Now, solving for the constants in the above equation, we find:

$$\begin{aligned} a_0 &= 1 = A + C \\ a_1 &= \tfrac{17}{4} = 2A + 2B - 2C - 2D + \frac{1}{4} \\ a_2 &= 30 = 4A + 8B + 4C + 8D + 2 \\ a_3 &= 41 = 8A + 24B - 8C - 24D + 9 \end{aligned}$$

which yields $A=1$, $B=2$, $C=0$, $D=1$. So, the final solution to the given recurrence is:

$$a_n = 2^n + 2n2^n + n(-2)^n + \frac{1}{8}n^2 2^n = \left[1 + 2n + \frac{1}{8}n^2\right]2^n + n(-2)^n, \quad n \geqslant 0.$$

4. **(a)** Let $A = \mathbb{Z} \times \mathbb{Z}$, and λ a fixed (constant) positive integer. Define two operations $\oplus$ and $\odot$ on A as

$$\begin{aligned} (a,b) \oplus (c,d) &= (a+c, b+d), \\ (a,b) \odot (c,d) &= (ad+bc, bd+\lambda ac). \end{aligned}$$

A is a commutative ring with identity under these two operations. You do not have to verify the ring axioms, but only mention what the additive and the multiplicative identities are in A (no need to prove their identity properties). Also, prove that A is an integral domain if and only if λ is ***not*** a perfect square. **(2 + 4)**

Solution Additive identity: $(0,0)$. Multiplicative identity: $(0,1)$.

Suppose that λ is not a perfect square, and $(a,b) \odot (c,d) = (0,0)$, that is, $ad + bc = 0$ and $bd + \lambda ac = 0$. But then, $a(bd + \lambda ac) - b(ad + bc) = 0$, that is, $(\lambda a^2 - b^2)c = 0$. Since λ is not a perfect square, we cannot have $\lambda a^2 - b^2 = 0$ or $\lambda = (b/a)^2$. Therefore we must have $c = 0$. This in turn implies $ad = 0$ and $bd = 0$. If $d = 0$, we have $c = d = 0$, whereas if $d \neq 0$, we have $a = b = 0$. That is, A does not contain non-zero zero divisors.

Conversely, let $\lambda = \alpha^2$. As derived above, we see that $\lambda a^2 - b^2 = 0$ is a necessary condition for the existence of non-zero zero divisors. We need to show that this condition is also sufficient. Taking $a = 1$ and $b = \alpha$ satisfies the condition. We should also have $ad + bc = 0$, that is, $\frac{a}{b} = -\frac{c}{d}$, that is, we can take $c = 1$ and $d = -\alpha$. But then, $bd + \lambda ac = -\alpha^2 + \lambda = 0$. Since $(1,\alpha)$ and $(1,-\alpha)$ are non-zero elements of A, and $(1,\alpha) \odot (1,-\alpha) = (0,0)$, A is not an integral domain.

(b) Let $(G,\circ)$ be a group, and c a fixed element of G. Define a binary operation $*$ on G by $a * b = a \circ c \circ b$ for all $a, b \in G$. Prove that $(G,*)$ is a group, clearly showing that all the properties of a group are satisfied. **(4)**

Solution $(G,*)$ is a group, because it satisfies the following properties of a group.

Closure: For any $p, q \in G$, $p * q = p \circ c \circ q \in G$, since $c \in G$ and G is closed under the operation $\circ$.

Associativity: For any $p, q, r \in G$, since G is associative under the operation $\circ$, we get:

$$(p * q) * r = (p \circ c \circ q) \circ c \circ r = p \circ c \circ (q \circ c \circ r) = p * (q * r)$$

Identity: c^{-1} is the identity element. For any element $p \in G$, we get:

$$p * c^{-1} = p \circ c \circ c^{-1} = p \circ e_G = p \quad \text{and} \quad c^{-1} * p = c^{-1} \circ c \circ p = e_G \circ p = p$$

where, $e_G \in G$ is the identity element with respect to the group $(G,\circ)$.

Inverse: For any element $p \in G$, let $p' \in G$ be its inverse with respect to $*$. Now, by definition we should get $p * p' = c^{-1} = p' * p$.

$$\therefore p \circ c \circ p' = c^{-1} \quad \text{or} \quad p' \circ c \circ p = c^{-1} \quad \Rightarrow \quad p' = c^{-1} \circ p^{-1} \circ c^{-1}$$

where, p^{-1} is the inverse of p with respect to the operation $\circ$.
