CS21201 Discrete Structures, Autumn 2021–2022

First Test

Date: Sep 07, 2021 | Time: 10:15am–11:30am | Maximum marks: 40

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1. Prove the following identity for all positive integers n.

$$\binom{2n}{n} = C(n) + \sum_{k=0}^{n-1} \binom{2n-2k-1}{n-k} C(k).$$

Here, $C(r)$ is the r-th Catalan number, and $\binom{s}{t} = \frac{s!}{t!(s-t)!}$ is the binomial coefficient s-choose-t. **(10)**

Solution We consider all paths from $(0,0)$ to (n,n) in the grid with R and U movements only. The total count of all such paths is $\binom{2n}{n}$, whereas $C(n)$ denotes the count of all valid paths (that is, paths that never cross the line $y = x$). So the sum on the right side should count the invalid paths from $(0,0)$ to (n,n). Take any such invalid path. By definition, this path crosses the line $y = x$ at least once. Let $k \in \{0,1,2,\ldots,n-1\}$ be the first instance when the path goes from (k,k) to $(k,k+1)$. This being the first instance, the part of the path from $(0,0)$ to (k,k) is a valid one, and the count of such paths is $C(k)$. We have a U movement from (k,k) to $(k,k+1)$. After this, we can choose any path from $(k,k+1)$ to (n,n), and the count of such paths is $\binom{(n-k)+(n-k-1)}{n-k}$. Finally, vary k in the allowed range $0 \leqslant k \leqslant n-1$.

2. From the semester examination results, the head of the CSE Department makes the following observations.

F_1 : *If Andy is meritorious, then Bob is not studious or Paul is not attentive.*

F_2 : *If Paul is attentive, then Liya is exceptional.*

F_3 : *If Liya is exceptional and Bob is studious, then Andy is meritorious.*

F_4 : *Bob is studious.*

You task is to answer the question: "*Is Paul attentive?*"

Frame the arguments logically, and formally deduce (applying logical inferencing rules) the answer being asked here. Present your solution as indicated in the following parts.

(a) Write all the propositions (that you use) with English statements (meaning). **(2)**

Solution We may use the following propositions.

am :	Andy is meritorious.	pa :	Paul is attentive.
bs :	Bob is studious.	le :	Liya is exceptional.

(b) Build a suitable propositional logic formula to encode each of the *four* statements F_1–F_4 given above. **(4)**

Solution The proposition logic encodings are as follows.

F_1 :	$am \to (\neg bs \vee \neg pa)$	F_3 :	$(le \wedge bs) \to am$
F_2 :	$pa \to le$	F_4 :	bs

We *claim* that we shall derive the goal $G : \neg pa$.

(c) Use logical inferencing rules to derive the answer, and conclude whether Paul is attentive or not. **(4)**

Solution The logical deduction procedure is given in the following.

$$\begin{array}{rcl} F_3 & : & (le \wedge bs) \to am \\ F_4 & : & bs \\ \hline \therefore F' & : & le \to am \end{array} \qquad \begin{array}{rcl} F_1 & : & am \to (\neg bs \vee \neg pa) \\ F_4 & : & bs \\ \hline \therefore F'' & : & am \to \neg pa \end{array} \qquad \begin{array}{rcl} F' & : & le \to am \\ F'' & : & am \to \neg pa \\ \hline \therefore F''' & : & le \to \neg pa \end{array} \qquad \begin{array}{rcl} F_2 & : & pa \to le \\ F''' & : & le \to \neg pa \\ \hline \therefore G & : & \neg pa \end{array}$$

Conclusion: *Paul is not attentive.*

3. Consider the following statements.

F_1 : *Tony and Mike are fans of Sachin Tendulkar.*

F_2 : *Every fan of Sachin Tendulkar is either a sports-lover, or a commentator, or both.*

F_3 : *No commentator likes rain during matches.*

F_4 : *All sports-lover likes high-scoring games.*

F_5 : *Mike dislikes whatever Tony likes and likes whatever Tony dislikes.*

F_6 : *Tony likes rain during matches and high-scoring games.*

Your tasks are to do the following.

(a) Write all the predicates (that you use) with English statements (meaning). **(2)**

Solution We may use the following predicates.

$fan(x)$: x is a fan of Sachin Tendulkar.

$comment(x)$: x is a commentator.

$splove(x)$: x is a sports-lover.

$likes(x,y)$: x likes y.

(b) Encode the above statements in predicate (first-order) logic. **(3)**

Solution The predicate (first-order) logic encodings are as follows.

$$F_1: \; fan(Tony) \wedge fan(Mike)$$

$$F_2: \; \forall x \big[fan(x) \rightarrow (splove(x) \vee comment(x)) \big]$$

$$F_3: \; \forall x \big[comment(x) \rightarrow \neg like(x, MRain) \big]$$

$$F_4: \; \forall x \big[splove(x) \rightarrow likes(x, HScore) \big]$$

$$F_5: \; \forall x \Big[\big(likes(Tony,x) \rightarrow \neg likes(Mike,x) \big) \wedge \big(\neg likes(Tony,x) \rightarrow likes(Mike,x) \big) \Big]$$

$$F_6: \; likes(Tony, MRain) \wedge likes(Tony, HScore)$$

(c) Use logical inferencing techniques (deduction rules) to prove that

G : "*There is a fan of Sachin Tendulkar who is a commentator, but not a sports-lover.*" **(5)**

Solution The goal statement can be encoded as follows.

$$G: \exists x \big[fan(x) \wedge comment(x) \wedge \neg splove(x) \big] \quad \Rightarrow \quad \neg G: \forall x \big[\neg fan(x) \vee \neg comment(x) \vee splove(x) \big]$$

Now, $(F_1 \wedge F_2 \wedge F_3 \wedge F_4 \wedge F_5 \wedge F_6 \rightarrow G)$ is valid $\Rightarrow$ $(F_1 \wedge F_2 \wedge F_3 \wedge F_4 \wedge F_5 \wedge F_6 \wedge \neg G)$ is unsatisfiable.

Eliminating $\forall x$ quantifier (as it implicitly denotes wherever x appears), the clauses formed by the above formulas are:

C_{11} : $fan(Tony)$

C_{12} : $fan(Mike)$

C_2 : $\neg fan(x) \vee splove(x) \vee comment(x)$

C_3 : $\neg comment(x) \vee \neg like(x, MRain)$

C_4 : $\neg splove(x) \vee likes(x, HScore)$

C_{51} : $\neg likes(Tony,x) \vee \neg likes(Mike,x)$

C_{52} : $likes(Tony,x) \vee likes(Mike,x)$

C_{61} : $likes(Tony, MRain)$

C_{62} : $likes(Tony, HScore)$

$C_{\neg G}$: $\neg fan(x) \vee \neg comment(x) \vee splove(x)$

The deduction procedure is given in the following.

$$\begin{array}{rcl} C_2 & : & \neg fan(x) \vee splove(x) \vee comment(x) \\ C_{\neg G} & : & \neg fan(x) \vee \neg comment(x) \vee splove(x) \\ \hline \therefore D_1 & : & \neg fan(x) \vee splove(x) \end{array} \qquad \begin{array}{rcl} C_{51} & : & \neg likes(Tony,x) \vee \neg likes(Mike,x) \\ C_{62} & : & likes(Tony, HScore) \\ \hline \therefore D_2 & : & \neg likes(Mike, HScore) \end{array}$$

$$\begin{array}{rcl} C_4 & : & \neg splove(x) \vee likes(x, HScore) \\ D_2 & : & \neg likes(Mike, HScore) \\ \hline \therefore D_3 & : & \neg splove(Mike) \end{array} \qquad \begin{array}{rcl} D_1 & : & \neg fan(x) \vee splove(x) \\ D_3 & : & \neg splove(Mike) \\ \hline \therefore D_4 & : & \neg fan(Mike) \end{array} \qquad \begin{array}{rcl} C_{12} & : & fan(Mike) \\ D_4 & : & \neg fan(Mike) \\ \hline \therefore & : & \perp \quad (contradiction) \end{array}$$

4. All the integers in the sequence 2021, 20821, 208821, 2088821, 20888821, ... (with any non-negative number of occurrences of the digit 8 between 20 and 21) are divisible by a common prime p. Find p, and prove the assertion. No credits without a valid proof. You may use $45^2 = 2025$ for simplifying your calculations. **(10)**

Solution The n-th integer in the sequence is $a_n = 208^n21$, where 8^n is the n-fold repetition of the digit 8. We show by induction on n that a_n is divisible by 47 for all $n \geqslant 0$.

[Basis] $a_0 = 2021 = 2025 - 4 = 45^2 - 2^2 = (45-2)(45+2) = 43 \times 47$.

[Induction] Suppose that a_n is divisible by 47. We have

$$\begin{aligned} a_{n+1} &= 20888\ldots821, \\ a_n &= 2088\ldots821, \\ a_{n+1} - a_n &= 18800\ldots000. \end{aligned}$$

Since $188 = 47 \times 4$, the inductive step is established. (Also note that $a_{n+1} - a_n$ is not divisible by 43.)