MTH 215 Number Theory

End Semester Examination

Total marks: 50 April 27, 2002 Time: $3 + \epsilon$ hours

Part I

1. Let $p \in \mathbb{P}$.

(a) Let x be an integer not divisible by p. Show that there exist integers a, b satisfying $0 < |a| < \sqrt{p}$, $0 < |b| < \sqrt{p}$ and $a \equiv bx \pmod{p}$. **(2)**

(b) Show that p can be expressed in the form $a^2 + 3b^2$ with $a, b \in \mathbb{Z}$ if and only if $p = 3$ or $p \equiv 1 \pmod{3}$. **(3)**

2. Let p be an odd prime. Prove that:

(a) Every primitive root modulo p is a quadratic nonresidue modulo p. **(2)**

(b) Every quadratic nonresidue modulo p is a primitive root modulo p if and only if p is of the form $2^{2^n} + 1$ for some $n \in \mathbb{Z}_+$. **(3)**

3. For $n \in \mathbb{N}$ define $\mathcal{F}(n)$ to be the set of those integers $m \in \mathbb{N}$ such that $\phi(m) = n$. Show that for every $n \in \mathbb{N}$ we have $\sum_{m \in \mathcal{F}(n)} \mu(m) = 0$, where μ denotes the Möbius function. **(5)**

4. Prove that there are infinitely many solutions in positive integers of both the equations $x^2 - 2y^2 = 1$ and $x^2 - 2y^2 = -1$. **(5)**

Part II

5. Recall that an integer $n \in \mathbb{N}$ is called a *perfect number* if the sum $\sigma(n)$ of positive divisors of n equals $2n$. No odd perfect numbers are known. Explain the fallacy in the following example discovered by Descartes: **(5)**
Let $n = 198585576189 = 3^2 \times 7^2 \times 11^2 \times 13^2 \times 22021$. Then

$$\begin{aligned}
\sigma(n) &= (1 + 3 + 3^2)(1 + 7 + 7^2)(1 + 11 + 11^2)(1 + 13 + 13^2)(1 + 22021) \\
&= 13 \times 57 \times 133 \times 183 \times 22022 \\
&= 397171152378 = 2n\,.
\end{aligned}$$

6. Let F_n, $n \in \mathbb{N}$, denote the Fibonacci sequence. Show that $(F_n F_{n+3}, 2F_{n+1}F_{n+2}, F_{n+3}^2 - 2F_{n+1}F_{n+2})$ is a Pythagorean triple for every $n \in \mathbb{N}$. Is the triple primitive? **(3+2)**

7. Let $m \in \mathbb{N}$ with $m \geqslant 2$. Show that $\sigma(m)$ is a power of two if and only if m is a product of (pairwise) distinct Mersenne primes. (A Mersenne prime is a prime of the form $2^p - 1$, $p \in \mathbb{P}$.) **(5)**

8. Prove that if the number $(a^2 + b^2)/(1 + ab)$ with $a, b \in \mathbb{N}$ is a positive integer, then it is a perfect square. **(5)**

9. Demonstrate that there exist infinitely many powers of 2 the four most significant digits in the decimal representation of each of which are 2002, i.e., there exist infinitely many $n \in \mathbb{N}$ such that we have the decimal representation $2^n = 2002\ldots$. **(5)**

10. Prove that the equation $x^4 - x^2y^2 + y^4 = z^2$ does not have a solution for $(x, y, z) \in \mathbb{N}^3$ with $x \neq y$. **(5)**
