The Structure of 2-Factors in a Graph

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The study of perfect matchings in a graph is a classical topic in graph theory, with many fundamental results, structural as well as algorithmic. One of the earliest results in graph theory is Petersen's theorem that every 3-regular bridgeless graph has a perfect matching. Tutte gave a necessary and sufficient condition for a graph to have a perfect matching, and Edmonds gave the first polynomial-time algorithm to check this condition. These can be found in any standard text in graph theory.

A perfect matching is a particular case of an f-factor in a graph. Given a function f from the vertex set of a graph to non-negative integers, an f-factor is a spanning subgraph in which the degree of a vertex v is $f(v)$. A perfect matching is the case when $f(v)$ is 1 for all vertices v, and is also called a 1-factor. Tutte also gave a necessary and sufficient condition for a graph to have an f-factor, for any specified function f.

Unlike perfect matchings, there can be many non-isomorphic f-factors in a graph. While Tutte's theorem gives a necessary and sufficient condition for the existence of an f-factor, it says nothing about the structure of the f-factor. In this talk, we look at some results that say more about the structure of the f-factor, and consider some of the many unanswered questions. We concentrate on the case of 2-factors, that is, $f(v) = 2$ for all vertices v.

A 2-factor in a graph is just a collection of cycles in a graph such that every vertex is in exactly one cycle. The most well-studied case of this problem is when we require the 2-factor to be connected. This is just the Hamilton cycle problem. We will consider other variations here.

It follows from Petersen's theorem that every 3-regular bridgeless graph has a 2-factor. However, deciding whether it has a connected 2-factor, that is, a Hamilton cycle, is an NP-complete problem. What happens if we want a disconnected 2-factor, that is, a 2-factor that is *not* a Hamilton cycle? This is unknown in general, but it was shown in [3] that K_4 is the only planar 3-regular bridgeless graph that does not have a disconnected 2-factor. It is conjectured in [5] that all 3-regular bipartite graphs with this property can be obtained from $K_{3,3}$ and the Heawood graph using a simple operation. It is also known that there are no k-regular bipartite graphs with this property, for $k \geq 4$. Some of these results can be extended for other restrictions on the 2-factor, such as on the parity of the number of components [1].

Another question is to see whether conditions that imply the existence of a Hamilton cycle also give other 2-factors. It was observed in [6] that the visibility graph of a point set has a Hamilton cycle iff not all points are collinear. In [4], this was generalized to show that the visibility graph of a set of n points contains a specified 2-factor with k cycles iff there do not exist $n - k + 1$ collinear points, except for one special case in which each component of the 2-factor is a triangle, and the point set has a particular structure.

We will also consider generalizations of other conditions for Hamiltonicity, such as minimum degree conditions [2], to the case of general 2-factors.

References

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