

Section outline

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 e Boolean lattice
 e Boolean lattice structure
 e Boolean algebra
 e Additional Boolean algebra

properties • [Additional Boolean algebra](#page-87-0) [properties](#page-87-0)

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 $k = \{a, b, c\}$
 $k = \{a, b, c\}$ • A set *A* of elements: $A = \{a, b, c\}$

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- A set *A* of elements: $A = \{a, b, c\}$
- **ks:** $A = \{a, b, c\}$
 $\mathbb{N} = \{0, 1, 2, 3, ...\}$ or $\{1, 2, 3, ...\} = \mathbb{Z}^+$ Natural numbers: $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ or $\{1, 2, 3, \ldots\} = \mathbb{Z}^+$

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Empty set: $\varnothing = \{\}$ Natural numbers: $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ or $\{1, 2, 3, \ldots\} = \mathbb{Z}^+$
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 $S \in S$? [Russell's paradox] Natural numbers: $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ or $\{1, 2, 3, \ldots\} = \mathbb{Z}^+$
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Complement of union (De Morgan): *A* ∪ *B* = *A* ∩ *B*

• Complement of union (De Morgan): $\overline{A \cup B} = \overline{A} \cap \overline{B}$

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dersection (De Morgan): $\overline{A \cap B} = \overline{A} \cup \overline{B}$ **•** Complement of intersection (De Morgan): $\overline{A \cap B} = \overline{A} \cup \overline{B}$

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• Complement of union (De Morgan): $\overline{A \cup B} = \overline{A} \cap \overline{B}$

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(A) **•** Complement of intersection (De Morgan): $\overline{A \cap B} = \overline{A} \cup \overline{B}$

• Power set of *A*: $P(A)$

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Complement of union (De Morgan): *A* ∪ *B* = *A* ∩ *B*

 $x^2 + y^2 = 0$

ion (De Morgan): $\overline{A \cup B} = \overline{A} \cap \overline{B}$
 $x^2 + z^2 = \overline{A} \cup \overline{B}$
 $x^2 + y^2 = \overline{A} \cup \overline{B}$
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 $x^2 + y^2 = \overline{A} \cup \overline{B}$ • Complement of intersection (De Morgan): $\overline{A \cap B} = \overline{A} \cup \overline{B}$

- Power set of $A: \mathcal{P}(A)$
	- $P({a,b}) = {\emptyset, {a}, {b}, {b}, {a, b}}$

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• Complement of intersection (De Morgan): $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Power set of *A*: P(*A*)

 $P({a,b}) = {\emptyset, {a}, {b}, {b}, {a,b}}$

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ion (De Morgan): $\overline{A \cup B} = \overline{A} \cap \overline{B}$

(A)
 $\{A\}$, $\{b\}$, $\{a, b\}$ }
 $\{X_k \text{ is a partition of } A \text{ if } A = X_1 \cup ... \cup X_k \text{ and }$ Non-empty X_1,\ldots,X_k is a partition of A if $A=X_1\cup\ldots\cup X_k$ and $X_i \cap X_j = \varnothing \mid_{i \neq j}$

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Complement of union (De Morgan): *A* ∪ *B* = *A* ∩ *B*

• Complement of intersection (De Morgan): $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Power set of *A*: P(*A*)

 $P({a,b}) = {\emptyset, {a}, {b}, {b}, {a,b}}$

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(A)
 $\{A\}$, $\{b\}$, $\{a, b\}$ }
 $\{X_k \text{ is a partition of } A \text{ if } A = X_1 \cup ... \cup X_k \text{ and } B \text{ and } \overline{A \cup B} \text{ constitute a partition of } U$ Non-empty X_1,\ldots,X_k is a partition of A if $A=X_1\cup\ldots\cup X_k$ and $X_i \cap X_j = \varnothing \mid_{i \neq j}$
	- $A \cap \overline{B}$, $B \cap \overline{A}$, $A \cap B$ and $\overline{A \cup B}$ constitute a partition of *U*

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Set algebra

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• Tuple: $\langle a, b \rangle$, $\langle 4, b, \alpha \rangle$

- **•** Tuple: $\langle a, b \rangle$, $\langle 4, b, \alpha \rangle$
- $\langle A \rangle$
 $\langle A \times B = \{ \langle a, b \rangle \mid a \in A, b \in B \}$ **•** Cartesian product: $A \times B = \{ \langle a, b \rangle \mid a \in A, b \in B \}$

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 $A \equiv \mathbf{1} \times A \equiv \mathbf{1}$

- **•** Tuple: $\langle a, b \rangle$, $\langle 4, b, \alpha \rangle$
- Cartesian product: $A \times B = \{ \langle a, b \rangle \mid a \in A, b \in B \}$
- $\langle A, \alpha \rangle$
 $\colon A \times B = \{ \langle a, b \rangle \mid a \in A, b \in B \}$
 $\{ \langle a, \alpha \rangle, \langle b, \alpha \rangle, \langle c, \alpha \rangle, \langle a, \beta \rangle, \langle b, \beta \rangle, \langle c, \beta \rangle \}$ ${a, b, c} \times {\alpha, \beta} = {\langle a, \alpha \rangle, \langle b, \alpha \rangle, \langle c, \alpha \rangle, \langle a, \beta \rangle, \langle b, \beta \rangle, \langle c, \beta \rangle}$

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 $\colon A \times B = \{ \langle a, b \rangle \mid a \in A, b \in B \}$
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 $\gamma \ge 1 \}$ ● Cartesian product: $A \times B = \{ \langle a, b \rangle \mid a \in A, b \in B \}$ ${a, b, c} \times {\alpha, \beta} = {\langle a, \alpha \rangle, \langle b, \alpha \rangle, \langle c, \alpha \rangle, \langle a, \beta \rangle, \langle b, \beta \rangle, \langle c, \beta \rangle}$

 $\mathbb{N} \times \mathbb{N} = \{ \langle i, j \rangle \mid i, j \geq 1 \}$

- **o** Tuple: $\langle a, b \rangle$, $\langle 4, b, \alpha \rangle$
- $R(A)$
 $\therefore A \times B = \{(a, b) \mid a \in A, b \in B\}$
 $\{(a, \alpha), \langle b, \alpha \rangle, \langle c, \alpha \rangle, \langle a, \beta \rangle, \langle b, \beta \rangle, \langle c, \beta \rangle\}$
 $\exists \geq 1\}$

Don sets A and B: $B \subseteq A \times B$ ● Cartesian product: $A \times B = \{ \langle a, b \rangle \mid a \in A, b \in B \}$ ${a, b, c} \times {\alpha, \beta} = {\langle a, \alpha \rangle, \langle b, \alpha \rangle, \langle c, \alpha \rangle, \langle a, \beta \rangle, \langle b, \beta \rangle, \langle c, \beta \rangle}$ $\mathbb{N} \times \mathbb{N} = \{ \langle i, j \rangle \mid i, j \geq 1 \}$
- Binary relation R on sets *A* and *B*: *R* ⊆ *A* × *B*

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- **o** Tuple: $\langle a, b \rangle$, $\langle 4, b, \alpha \rangle$
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 $\exists \geq 1\}$

Din sets A and B: $R \subseteq A \times B$

tion of R: $\chi_R(a, b) = \begin{cases} 1 & \text{if } \langle a, b \rangle \in R \\ 0 & \text{if } \langle a, b \rangle \notin R \end{cases}$ \bullet Cartesian product: *A* × *B* = { $\langle a, b \rangle$ | *a* ∈ *A*, *b* ∈ *B*} ${a, b, c} \times {\alpha, \beta} = {\langle a, \alpha \rangle, \langle b, \alpha \rangle, \langle c, \alpha \rangle, \langle a, \beta \rangle, \langle b, \beta \rangle, \langle c, \beta \rangle}$ $\mathbb{N} \times \mathbb{N} = \{ \langle i, j \rangle \mid i, j \geq 1 \}$
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- **o** Tuple: $\langle a, b \rangle$, $\langle 4, b, \alpha \rangle$
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- Binary relation R on sets *A* and *B*: *R* ⊆ *A* × *B*
- Characteristic function of \mathcal{R} : $\chi_{\mathcal{R}}(a,b) = \begin{cases} 1 & \text{if } \langle a,b \rangle \in \mathcal{R} \\ 0 & \text{if } \langle a,b \rangle \notin \mathcal{R} \end{cases}$ 0 if $\langle a, b \rangle \notin \mathcal{R}$
- R ⊆ *A* × *A* is reflexive if ∀*x* ∈ *A*. *x*R*x*

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- **o** Tuple: $\langle a, b \rangle$, $\langle 4, b, \alpha \rangle$
- $R(A)$
 $\therefore A \times B = \{(a, b) \mid a \in A, b \in B\}$
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- R ⊆ *A* × *A* is reflexive if ∀*x* ∈ *A*. *x*R*x*
- R ⊆ *A* × *A* is symmetric if ∀*x*, *y* ∈ *A*. *x*R*y* ⇒ *y*R*x*

 $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A} \subseteq \mathcal{B} \Rightarrow$

- **o** Tuple: $\langle a, b \rangle$, $\langle 4, b, \alpha \rangle$
- $R(A)$
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- R ⊆ *A* × *A* is reflexive if ∀*x* ∈ *A*. *x*R*x*
- R ⊆ *A* × *A* is symmetric if ∀*x*, *y* ∈ *A*. *x*R*y* ⇒ *y*R*x*
- R ⊆ *A* × *A* is transitive if ∀*x*, *y*, *z* ∈ *A*. *x*R*y* ∧ *y*R*z* ⇒ *x*R*z*

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- **o** Tuple: $\langle a, b \rangle$, $\langle 4, b, \alpha \rangle$
- [Relations](#page-29-0)
 $R(A \times B = \{(a, b) \mid a \in A, b \in B\}$
 $\{(a, \alpha), \langle b, \alpha \rangle, \langle c, \alpha \rangle, \langle a, \beta \rangle, \langle b, \beta \rangle, \langle c, \beta \rangle\}$
 $\{a \in A \text{ and } B: B \subseteq A \times B$
 $\{b \in B \text{ from } B: \chi_{\mathcal{R}}(a, b) = \begin{cases} 1 & \text{if } \langle a, b \rangle \in \mathcal{R} \\ 0 & \text{if } \langle a, b \rangle \notin \mathcal{R} \end{cases}$
 $\{x \in A : x \$ **•** Cartesian product: $A \times B = \{ \langle a, b \rangle \mid a \in A, b \in B \}$ ${a, b, c} \times {\alpha, \beta} = {\langle a, \alpha \rangle, \langle b, \alpha \rangle, \langle c, \alpha \rangle, \langle a, \beta \rangle, \langle b, \beta \rangle, \langle c, \beta \rangle}$ $\mathbb{N} \times \mathbb{N} = \{ \langle i, j \rangle \mid i, j \geq 1 \}$
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- R ⊆ *A* × *A* is reflexive if ∀*x* ∈ *A*. *x*R*x*
- R ⊆ *A* × *A* is symmetric if ∀*x*, *y* ∈ *A*. *x*R*y* ⇒ *y*R*x*
- R ⊆ *A* × *A* is transitive if ∀*x*, *y*, *z* ∈ *A*. *x*R*y* ∧ *y*R*z* ⇒ *x*R*z*
- **•** $R \subset A \times A$ is antisymmetric if $\forall x, y \in A$. $x \mathcal{R} y \wedge y \mathcal{R} x \Rightarrow x = y$

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- **o** Tuple: $\langle a, b \rangle$, $\langle 4, b, \alpha \rangle$
- $R(\mathbf{A} \times \mathbf{B}) = \{ \langle \mathbf{a}, \mathbf{b} \rangle \mid \mathbf{a} \in \mathbf{A}, \mathbf{b} \in \mathbf{B} \}$
 $\{ \langle \mathbf{a}, \alpha \rangle, \langle \mathbf{b}, \alpha \rangle, \langle \mathbf{c}, \alpha \rangle, \langle \mathbf{a}, \beta \rangle, \langle \mathbf{b}, \beta \rangle, \langle \mathbf{c}, \beta \rangle \}$
 $\{ \beta \geq 1 \}$

Don sets A and B: $R \subseteq A \times B$

tion of $\mathcal{R}: \chi_{\mathcal{$ **•** Cartesian product: $A \times B = \{ \langle a, b \rangle \mid a \in A, b \in B \}$ ${a, b, c} \times {\alpha, \beta} = {\langle a, \alpha \rangle, \langle b, \alpha \rangle, \langle c, \alpha \rangle, \langle a, \beta \rangle, \langle b, \beta \rangle, \langle c, \beta \rangle}$ $\mathbb{N} \times \mathbb{N} = \{ \langle i, j \rangle \mid i, j \geq 1 \}$
- Binary relation R on sets *A* and *B*: *R* ⊆ *A* × *B*
- Characteristic function of \mathcal{R} : $\chi_{\mathcal{R}}(a,b) = \begin{cases} 1 & \text{if } \langle a,b \rangle \in \mathcal{R} \\ 0 & \text{if } \langle a,b \rangle \notin \mathcal{R} \end{cases}$ 0 if $\langle a, b \rangle \notin \mathcal{R}$
- R ⊆ *A* × *A* is reflexive if ∀*x* ∈ *A*. *x*R*x*
- R ⊆ *A* × *A* is symmetric if ∀*x*, *y* ∈ *A*. *x*R*y* ⇒ *y*R*x*
- R ⊆ *A* × *A* is transitive if ∀*x*, *y*, *z* ∈ *A*. *x*R*y* ∧ *y*R*z* ⇒ *x*R*z*
- **•** $R \subset A \times A$ is antisymmetric if $\forall x, y \in A$. $x \mathcal{R} y \wedge y \mathcal{R} x \Rightarrow x = y$
- **•** Equivalence relation: R is reflexive, symmetric and transitive

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目

- **o** Tuple: $\langle a, b \rangle$, $\langle 4, b, \alpha \rangle$
- $R = \{ \langle a, b \rangle \mid a \in A, b \in B \}$
 $\{ \langle a, \alpha \rangle, \langle b, \alpha \rangle, \langle c, \alpha \rangle, \langle a, \beta \rangle, \langle b, \beta \rangle, \langle c, \beta \rangle \}$
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Relations (contd.)

Connected relation: ∀*x*, *y* ∈ *A*, either *x*R*y* or *y*R*x*

Relations (contd.)

- Connected relation: ∀*x*, *y* ∈ *A*, either *x*R*y* or *y*R*x*
- Total order: Connected partial order (eg \preceq on \mathbb{R})

Relations (contd.)

- **konductives [Relations](#page-35-0)**

 λ
 $\forall x, y \in A$, either $x \mathcal{R} y$ or $y \mathcal{R} x$

coted partial order (eg \preceq on \mathbb{R})
 $\forall x \in A, \langle x, x \rangle \notin \mathcal{R}$ Connected relation: ∀*x*, *y* ∈ *A*, either *x*R*y* or *y*R*x*
- Total order: Connected partial order (eq \prec on \mathbb{R})
- \bullet Irreflexive relation: $\forall x \in A, \langle x, x \rangle \notin \mathcal{R}$

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 (5.7) (6.7)
- **konductives [Relations](#page-36-0)**

∴ ∀x, *y* ∈ *A*, either *xRy* or *yRx*

ccted partial order (eg \preceq on ℝ)

∀x ∈ *A*, $\langle x, x \rangle \notin \mathcal{R}$

n: $\langle x, y \rangle \in \mathcal{R} \Rightarrow \langle y, x \rangle \notin \mathcal{R}$ Connected relation: ∀*x*, *y* ∈ *A*, either *x*R*y* or *y*R*x*
- Total order: Connected partial order (eq \prec on \mathbb{R})
- \bullet Irreflexive relation: $\forall x \in A, \langle x, x \rangle \notin \mathcal{R}$
- Asymmetric relation: $\langle x, y \rangle \in \mathcal{R} \Rightarrow \langle y, x \rangle \notin \mathcal{R}$

 $1.7.1 \times 1.7.1$

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Relations (contd.)

- Connected relation: ∀*x*, *y* ∈ *A*, either *x*R*y* or *y*R*x*
- Total order: Connected partial order (eq \prec on \mathbb{R})
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- **kofficial cofficial order** (eg \preceq on ℝ)

∴ ∀x, y ∈ A, either x \mathcal{R} y or y \mathcal{R} x

coted partial order (eg \preceq on ℝ)

∴ ∀x ∈ A, $\langle x, x \rangle \notin \mathcal{R}$

correflexive and transitive (∴ asymmetric)

correflexive and Strict order: R is irreflexive and transitive (∴ asymmetric)

- Connected relation: ∀*x*, *y* ∈ *A*, either *x*R*y* or *y*R*x*
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- Strict order: R is irreflexive and transitive (∴ asymmetric)
- **kofficial structures [Relations](#page-38-0)**
 kofficial order (eg \preceq on \mathbb{R})
 $\forall x, y \in A$, either $x \mathcal{R} y$ or $y \mathcal{R} x$

coted partial order (eg \preceq on \mathbb{R})
 $\forall x \in A, \langle x, x \rangle \notin \mathcal{R}$

correflexive and transitiv If is a PO on *A*, then <: *x* < *y* ≡ *x y* ∧ *x* 6= *y* is a SO on *A*

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 $\left\{ \begin{array}{ccc} \square & \rightarrow & \leftarrow & \square \end{array} \right.$

- Connected relation: ∀*x*, *y* ∈ *A*, either *x*R*y* or *y*R*x*
- Total order: Connected partial order (eq \prec on \mathbb{R})
- \bullet Irreflexive relation: $\forall x \in A, \langle x, x \rangle \notin \mathcal{R}$
- Asymmetric relation: $\langle x, y \rangle \in \mathcal{R} \Rightarrow \langle y, x \rangle \notin \mathcal{R}$
- Strict order: R is irreflexive and transitive (∴ asymmetric)
- **koffinal conducts [Relations](#page-39-0)**

∴ ∀x, *y* ∈ *A*, either *xRy* or *yRx*

coted partial order (eg \preceq on \mathbb{R})

∴ ∀*x* ∈ *A*, $\langle x, x \rangle \notin \mathcal{R}$

conducts and transitive (∴ asymmetric)

then $\prec: x \prec y \equiv x \preceq y \land x \neq y$ If is a PO on *A*, then <: *x* < *y* ≡ *x y* ∧ *x* 6= *y* is a SO on *A*
- **•** If \lt is a SO on A, then \lt : $x \lt y \le x \lt y \vee x = y$ is a PO on A

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- Connected relation: ∀*x*, *y* ∈ *A*, either *x*R*y* or *y*R*x*
- Total order: Connected partial order (eq \prec on \mathbb{R})
- \bullet Irreflexive relation: $\forall x \in A, \langle x, x \rangle \notin \mathcal{R}$
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- **•** If \prec is a PO on *A*, then $\lt:$: $x \lt y \triv x \lt y \wedge x \neq y$ is a SO on *A*
- **•** If \lt is a SO on A, then \lt : $x \lt y \le x \lt y \vee x = y$ is a PO on A

^N^I ^D ^A^I \mathbb{Z} **R ^G^P^UR** \sim **[Relations](#page-40-0)**
 Relations
 Suppose $\langle A, \preceq \rangle$ is a poset, *M* ∈ *A* (*m* ∈ *A*), *S* ⊂ *A M* (*m*) is a maximal (minimal) element of *S* iff $M \in S$ ($m \in S$) and $\exists x \in S$ st $M < x$ ($x < m$) *M* (*m*) is a maximum (minimum) of *S* iff *M* ∈ *S* (*m* ∈ *S*) and $\forall x \in S$, $x \leq M$ (*m* $\leq x$)

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- Connected relation: ∀*x*, *y* ∈ *A*, either *x*R*y* or *y*R*x*
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^N^I ^D ^A^I \mathbb{Z} **R ^G^P^UR** \sim --yog, km
s kOflm^**[Discrete structures](#page-41-0) [Relations](#page-41-0)** Suppose $\langle A, \preceq \rangle$ is a poset, *M* ∈ *A* (*m* ∈ *A*), *S* ⊂ *A M* (*m*) is a maximal (minimal) element of *S* iff $M \in S$ ($m \in S$) and $\exists x \in S$ st $M < x$ ($x < m$) *M* (*m*) is a maximum (minimum) of *S* iff *M* ∈ *S* (*m* ∈ *S*) and $\forall x \in S$, $x \leq M$ (*m* $\leq x$)

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Lattices

Let $\langle A, \prec \rangle$ be a poset, let *x*, $\gamma \in A$

- The *meet* of *x* and *y* (*x* ∧ *y*), is the maximum of all lower bounds for *x* and *y*: $x \wedge y = \max \{ w \in A : w \leq x, w \leq y \}$, *glb* for *x* and *y*
- **Example 3 [Lattices](#page-42-0)**
 Example 3 Lattices
 Example 3 Lattices
 Example 3 Let X **,** $W \leq Y$ **,** W **and** W **
** \forall $(X \vee Y)$ **, is the minimum of all upper bounds for
** \forall $(X \vee Y)$ **, is the minimum of all upper bounds** The *join* of *x* and *y* (*x* ∨ *y*), is the minimum of all upper bounds for *x* and *y*; $x \vee y = \min \{ z \in A : x \prec z, y \prec z \}$, *lub* for *x* and *y*

A poset $\langle A, \prec \rangle$ is a lattice iff every pair of elements in *A* have both a meet and a join

^N^I ^D ^A^I \mathbb{Z} **R ^G^P^UR** \sim --yog, km
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 $\left\{ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right.$

Lattices

Let $\langle A, \preceq \rangle$ be a poset, let *x*, $y \in A$

- The *meet* of *x* and *y* (*x* ∧ *y*), is the maximum of all lower bounds for *x* and *y*: $x \wedge y = \max \{ w \in A : w \leq x, w \leq y \}$, glb for *x* and *y*
- The *join* of *x* and *y* (*x* ∨ *y*), is the minimum of all upper bounds for *x* and *y*; $x \vee y = \min\{z \in A : x \leq z, y \leq z\}$, *lub* for *x* and *y*

A poset $\langle A, \prec \rangle$ is a lattice iff every pair of elements in A have both a meet and a join

Basic order properties of meet and join

$$
\bullet \ \ x \wedge y \preceq \{x,y\} \preceq x \vee y
$$

•
$$
x \preceq y
$$
 iff $x \wedge y = x$

•
$$
x \preceq y
$$
 iff $x \vee y = y$

- If *x y*, then *x* ∧ *z y* ∧ *z* and *x* ∨ *z y* ∨ *z*
- If *x y* and *z w*, then *x* ∧ *z y* ∧ *w* and *x* ∨ *z y* ∨ *w*

Theorem

If $x \prec y$, then $x \wedge z \prec y \wedge z$ and $x \vee z \prec y \vee z$

Proof.

- Let *v* = *x* ∧ *z* and *u* = *y* ∧ *z*
- By transitivity, *v* is a lb for *y* and *z*
- **Conserver structures** [Lattices](#page-44-0)

of meet and join
 $x \vee y$
 $z \preceq y \wedge z$ and $x \vee z \preceq y \vee z$
 x , then $x \wedge z \preceq y \wedge w$ and $x \vee z \preceq y \vee w$
 $x \wedge z$ and $x \vee z \preceq y \vee z$
 $x = y \wedge z$
 $x \triangleleft y$ and z
 $x \preceq u$ (as v is t By definition of ∧, *v u* (as *v* is the maximum among all lbs)

Similarly, the other clause may be proven

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x ◦ ◦ *v*

◦ *u* ◦*z*

[Discrete structures](#page-45-0)
 $= y \land x, \quad x \lor y = y \lor x$
 $\land z = x \land (y \land z), \quad (x \lor y) \lor z = x \lor (y \lor z)$
 \land) = x, $x \lor (x \land y) = x$

x, $x \lor x = x$
 $\lor \preceq x [x \land y \preceq \{x, y\} \text{ applied twice}]$

ansitivity of \preceq] **Commutativity** $x \wedge y = y \wedge x$, $x \vee y = y \vee x$ Associativity $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$ **Absorption** $x \wedge (x \vee y) = x$, $x \vee (x \wedge y) = x$ **Idempotence** $x \wedge x = x$, $x \vee x = x$

Associativity of meet.

- \bullet (*x* ∧ *y*) ∧ *z* \le *x* ∧ *y* \le *x* [*x* ∧ *y* \le {*x*, *y*} applied twice]
- \bullet (*x* ∧ *y*) ∧ *z* \prec *x* [transitivity of \prec]

[Discrete structures](#page-46-0)
 $\begin{aligned}\n&= y \wedge x, \quad x \vee y = y \vee x \\
&\wedge z = x \wedge (y \wedge z), \quad (x \vee y) \vee z = x \vee (y \vee z) \\
y) = x, \quad x \vee (x \wedge y) = x \\
x, \quad x \vee x = x \\
\therefore \\
y \leq x [x \wedge y \leq \{x, y\} \text{ applied twice}] \\
\text{ansitivity of } \preceq] \\
&\leq \{x, y\} \\
z \text{ [If } x \preceq y, \text{ then } x \wedge z \preceq y]\n\end{aligned}$ **Commutativity** $x \wedge y = y \wedge x$, $x \vee y = y \vee x$ Associativity $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$ **Absorption** $x \wedge (x \vee y) = x$, $x \vee (x \wedge y) = x$ **Idempotence** $x \wedge x = x$, $x \vee x = x$

Associativity of meet.

•
$$
(x \land y) \land z \preceq x \land y \preceq x [x \land y \preceq \{x, y\}
$$
 applied twice]

•
$$
(x \land y) \land z \preceq x
$$
 [transitivity of \preceq]

$$
\bullet \ (x \wedge y) \preceq y [x \wedge y \preceq \{x, y\}]
$$

•
$$
(x \land y) \land z \preceq y \land z
$$
 [If $x \preceq y$, then $x \land z \preceq y$]

[Lattices](#page-47-0)
 $\begin{aligned}\n&= y \land x, \quad x \lor y = y \lor x \\
& \land z = x \land (y \land z), \quad (x \lor y) \lor z = x \lor (y \lor z) \\
y) = x, \quad x \lor (x \land y) = x \\
x, \quad x \lor x = x\n\end{aligned}$
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&= x \land (y \land z) = x \land (y \land z) \\
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Associativity of meet.

- \bullet (*x* ∧ *y*) ∧ *z* \le *x* ∧ *y* \le *x* [*x* ∧ *y* \le {*x*, *y*} applied twice]
- \bullet (*x* ∧ *y*) ∧ *z* \prec *x* [transitivity of \prec]

$$
\bullet \ (x \wedge y) \preceq y [x \wedge y \preceq \{x, y\}]
$$

- \bullet (*x* ∧ *y*) ∧ *z* \prec *y* ∧ *z* [If *x* \prec *y*, then *x* ∧ *z* \prec *y*]
- Thus (*x* ∧ *y*) ∧ *z* is a lb of both *x* and *y* ∧ *z*
- ∴ (*x* ∧ *y*) ∧ *z x* ∧ (*y* ∧ *z*) [glb of *x* and *y* ∧ *z*]

[Lattices](#page-48-0)
 $\begin{aligned}\n&= y \land x, \quad x \lor y = y \lor x \\
\land z &= x \land (y \land z), \quad (x \lor y) \lor z = x \lor (y \lor z) \\
y) &= x, \quad x \lor (x \land y) = x \\
x, \quad x \lor x &= x\n\end{aligned}$
 $\begin{aligned}\n&= x \land (y \land z), \\
&= x \land (y \land z) = x \land (y \land z) \\
&= x \land (y \land z) = x \land (y \land z) \\
&= x \land (y \land z) = y \land (y \land z) \\
&= x \land (y \land z$ **Commutativity** $x \wedge y = y \wedge x$, $x \vee y = y \vee x$ Associativity $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$ **Absorption** $x \wedge (x \vee y) = x$, $x \vee (x \wedge y) = x$ **Idempotence** $x \wedge x = x$, $x \vee x = x$

Associativity of meet.

- \bullet (*x* ∧ *y*) ∧ *z* \le *x* ∧ *y* \le *x* [*x* ∧ *y* \le {*x*, *y*} applied twice]
- \bullet (*x* ∧ *y*) ∧ *z* \prec *x* [transitivity of \prec]

$$
\bullet \ (x \wedge y) \preceq y [x \wedge y \preceq \{x, y\}]
$$

- \bullet (*x* ∧ *y*) ∧ *z* \prec *y* ∧ *z* [If *x* \prec *y*, then *x* ∧ *z* \prec *y*]
- Thus (*x* ∧ *y*) ∧ *z* is a lb of both *x* and *y* ∧ *z*
- ∴ (*x* ∧ *y*) ∧ *z x* ∧ (*y* ∧ *z*) [glb of *x* and *y* ∧ *z*]
- Also, *x* ∧ (*y* ∧ *z*) (*x* ∧ *y*) ∧ *z* [on similar lines]

Example 3 [Lattices](#page-49-0)
 $\begin{aligned}\n&= y \land x, \quad x \lor y = y \lor x \\
\land z &= x \land (y \land z), \quad (x \lor y) \lor z = x \lor (y \lor z) \\
y) &= x, \quad x \lor (x \land y) = x \\
x, \quad x \lor x &= x\n\end{aligned}$
 $\begin{aligned}\n&= x \land (y \land z) \\
&= x \$ **Commutativity** $x \wedge y = y \wedge x$, $x \vee y = y \vee x$ Associativity $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$ **Absorption** $x \wedge (x \vee y) = x$, $x \vee (x \wedge y) = x$ **Idempotence** $x \wedge x = x$, $x \vee x = x$

Associativity of meet.

•
$$
(x \land y) \land z \preceq x \land y \preceq x [x \land y \preceq \{x, y\}
$$
 applied twice]

•
$$
(x \land y) \land z \preceq x
$$
 [transitivity of \preceq]

$$
\bullet \ (x \wedge y) \preceq y [x \wedge y \preceq \{x, y\}]
$$

•
$$
(x \land y) \land z \preceq y \land z
$$
 [If $x \preceq y$, then $x \land z \preceq y$]

- Thus (*x* ∧ *y*) ∧ *z* is a lb of both *x* and *y* ∧ *z*
- ∴ (*x* ∧ *y*) ∧ *z x* ∧ (*y* ∧ *z*) [glb of *x* and *y* ∧ *z*]

Also, *x* ∧ (*y* ∧ *z*) (*x* ∧ *y*) ∧ *z* [on similar lines]

∴ $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ [if $a \preceq b$ and $b \preceq a$ then $a = b$]

Absorbtion.

$$
\bullet \; x \preceq x \vee y \; [\{x,y\} \preceq x \vee y]
$$

∴ $x \wedge (x \vee y) = x [x \preceq y \text{ iff } x \wedge y = x]$

Absorbtion.

$$
\bullet \; x \preceq x \vee y \; [\{x,y\} \preceq x \vee y]
$$

$$
\therefore x \wedge (x \vee y) = x [x \preceq y \text{ iff } x \wedge y = x]
$$

Idempotence.

kOflm^**[Discrete structures](#page-51-0) [Lattices](#page-51-0)** *x* ∧ *x* = *x* ∧ (*x* ∨ (*x* ∧ *y*)) = *x* [Absorbtion, applied twice]

Principle of Duality

The dual of any theorem in a lattice is also a theorem.

biscrete structures [Lattices \(contd.\)](#page-52-0)
 $x \geq 0$
 x **Bounded lattice:** It has a maximum element (1) and a minimum element (0), in which case identity properties hold:

$$
0 \vee x = x = x \vee 0, \quad 1 \wedge x = x = x \wedge 1
$$

$$
0 \wedge x = 0 = x \wedge 0, \quad 1 \vee x = 1 = x \vee 1
$$

Every finite lattice is bounded

Bounded lattice: It has a maximum element (1) and a minimum element (0), in which case identity properties hold:

$$
0 \vee x = x = x \vee 0, \quad 1 \wedge x = x = x \wedge 1
$$

$$
0 \wedge x = 0 = x \wedge 0, \quad 1 \vee x = 1 = x \vee 1
$$

Every finite lattice is bounded

Distributive lattice: If $\forall x, y, z \in A$,

•
$$
x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)
$$
 and

$$
\bullet \; x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)
$$

Are these lattices distributive?

Bounded lattice: It has a maximum element (1) and a minimum element (0), in which case identity properties hold:

$$
0 \vee x = x = x \vee 0, \quad 1 \wedge x = x = x \wedge 1
$$

$$
0 \wedge x = 0 = x \wedge 0, \quad 1 \vee x = 1 = x \vee 1
$$

Every finite lattice is bounded

Distributive lattice: If $\forall x, y, z \in A$,

•
$$
x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)
$$
 and

$$
\bullet \; x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)
$$

Are these lattices distributive?

Is P(*A*) **for set** *A* **distributive?**

?
$$
x \cap (y \cup z) = (x \cap y) \cup (x \cap z)
$$
 and

$$
? \ x \cup (y \cap z) = (x \cup y) \cap (x \cup z)
$$

Complemented lattice

- *Complement in a bounded lattice: z* is the complement of *x* iff
	- *x* ∧ *z* = 0 and
	- *x* ∨ *z* = 1
- **• Bounded complemented lattice:** every element has a complement
- **attice**
 attice
 bounded lattice: z is the complement of x iff
 mented lattice: every element has a

libutive lattice with minimum 0 and maximum 1,

of elements are unique, provided they exist let

rements of x .. • In a bounded distributive lattice with minimum 0 and maximum 1, the complements of elements are unique, provided they exist – let \bar{x} and *z* be complements of *x* ...

Complemented lattice

- *Complement in a bounded lattice: z* is the complement of *x* iff
	- *x* ∧ *z* = 0 and
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- **Bounded complemented lattice:** every element has a complement
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 bounded lattice: z is the complement of x iff
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ements of x ...
 z) = • In a bounded distributive lattice with minimum 0 and maximum 1, the complements of elements are unique, provided they exist – let \bar{x} and *z* be complements of *x* ... 1 1

a

b

1

0

c

a

 $a \vee b$ $a \vee c$ *b* ∨ *c*

a b c

0

c

b

0

$$
\bullet \ \ \bar{x} = \bar{x} \wedge 1 = \bar{x} \wedge (x \vee z) =
$$

- \bullet $(\bar{x} \wedge x) \vee (\bar{x} \wedge z) =$
- \bullet 0 \vee ($\bar{x} \wedge z$) =
- \bullet $(x \wedge z) \vee (\overline{x} \wedge z) =$

$$
\bullet \ (x \vee \bar{x}) \wedge z = 1 \wedge z = z
$$

[Discrete structures](#page-57-0) **[Boolean lattice](#page-57-0)**

ded complemented distributive lattice

: Max: 1, Min: 0

s unique complement

apply **Boolean lattice:** Bounded complemented distributive lattice

- Extreme elements: Max: 1, Min: 0
- **•** Distributivity holds
- **•** Every element has unique complement
- De Morgan's laws apply

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Biscrete structures [Boolean lattice](#page-58-0)

Scolean lattice

∴ Max: 1, Min: 0

Sunique complement

apply
 A Boolean lattice $\langle A, \preceq, -, 0, 1 \rangle$ **Boolean lattice:** Bounded complemented distributive lattice

- Extreme elements: Max: 1, Min: 0
- **•** Distributivity holds
- Every element has unique complement
- De Morgan's laws apply

De Morgan's laws in a Boolean lattice $\langle A, \preceq, \text{-}, 0, 1 \rangle$

$$
\bullet \ \overline{x \wedge y} = \overline{x} \vee \overline{y}
$$

$$
\bullet \ \overline{x\vee y} = \overline{x} \wedge \overline{y}
$$

kOflm^**[Discrete structures](#page-59-0) [Boolean lattice](#page-59-0) Boolean lattice:** Bounded complemented distributive lattice

- Extreme elements: Max: 1, Min: 0
- **•** Distributivity holds
- Every element has unique complement
- De Morgan's laws apply

De Morgan's laws in a Boolean lattice $\langle A, \preceq, \text{-}, 0, 1 \rangle$

$$
\bullet \ \overline{x \wedge y} = \overline{x} \vee \overline{y}
$$

$$
\bullet \ \overline{x\vee y} = \overline{x} \wedge \overline{y}
$$

Meet of complements is 0

$$
\bullet \; (x \wedge y) \wedge (\overline{x} \vee \overline{y}) =(x \wedge y \wedge \overline{x}) \vee (x \wedge y \wedge \overline{y})
$$

$$
\bullet = 0 \lor 0 = 0
$$

Boolean lattice: Bounded complemented distributive lattice

- Extreme elements: Max: 1, Min: 0
- **•** Distributivity holds
- Every element has unique complement
- De Morgan's laws apply

De Morgan's laws in a Boolean lattice $\langle A, \preceq, \text{-}, 0, 1 \rangle$

Join of complements is 1

$$
\overline{y} \quad \overline{x \wedge y} = \overline{x} \vee \overline{y}
$$

² *x* ∨ *y* = *x* ∧ *y*

Meet of complements is 0

$$
\bullet (x \wedge y) \wedge (\overline{x} \vee \overline{y}) =(x \wedge y \wedge \overline{x}) \vee (x \wedge y \wedge \overline{y})
$$

 $\bullet = 0 \vee 0 = 0$

Discrete structures

\nBoolean lattice

\n: Max: 1, Min: 0

\nis unique complement

\napply

\n**a Boolean lattice**
$$
\langle \mathcal{A}, \preceq, \neg, 0, 1 \rangle
$$

\nJoin of complements is 1

\n**•** $(x \land y) \lor (\overline{x} \lor \overline{y}) = ((x \land y) \lor \overline{x}) \lor \overline{y}$

\n**•** $= ((x \lor \overline{x}) \land (y \lor \overline{x})) \lor \overline{y}$

\n**•** $= (1 \land (y \lor \overline{x})) \lor \overline{y}$

\n**•** $= (y \lor \overline{x}) \lor \overline{y}$

\n**•** $= \overline{x} \lor (y \lor \overline{y})$

$$
\bullet = ((x \vee \overline{x}) \wedge (y \vee \overline{x})) \vee \overline{y}
$$

$$
\bullet = (1 \land (y \lor \overline{x})) \lor \overline{y}
$$

$$
\bullet = (y \vee \overline{x}) \vee \overline{y}
$$

$$
\bullet = \overline{x} \vee (y \vee \overline{y})
$$

$$
\bullet=\overline{x}\vee 1=1
$$

- \bullet Let A be a lattice with min 0
- $a \in A$ is join irreducible if $a \neq x \vee y$ for $x, y \preceq a$, alternatively
	- *a* = *x* ∨ *y* implies *a* = *x* or *a* = *y*
- 0 is join irreducible

kOflm^**[Discrete structures](#page-61-0) [Boolean lattice structure](#page-61-0)** *a* 0 *a* b_1 0 *a c b*¹ *b*²

• If b_1 \leq *c* and b_2 \leq *c* (immediate preds) of *c* then $c = b_1 \vee b_2$

- \bullet Let A be a lattice with min 0
- $a \in \mathcal{A}$ is join irreducible if $a \neq x \vee y$ for $x, y \prec a$, alternatively
	- $a = x \vee y$ implies $a = x$ or $a = y$
- 0 is join irreducible

- **•** If $b_1 \nless c$ and $b_2 \nless c$ (immediate preds) of *c* then $c = b_1 \vee b_2$
- \bullet $a \neq 0$ is join irreducible if and only if *a* has a unique immediate predecessor

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- \bullet $a \neq 0$ is join irreducible if and only if *a* has a unique immediate predecessor
- Elements immediately succeeding 0 are atoms (join irreducible)

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- \bullet $a \neq 0$ is join irreducible if and only if *a* has a unique immediate predecessor
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- Any element *a* can be expressed as the join of a set of atoms

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- 0 is join irreducible

 $1.7.1 \times 1.7.1$

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- Not unique for non-distributive lattice (diamond lattice)

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- 0 is join irreducible

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- **•** For finite lattice $a = d_1 ∨ d_2 ∨ … ∨ d_n$, d_i are join irreducible

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•
$$
d_j = d_i \vee d_j
$$
 for $d_i \preceq d_j$

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- $a \in A$ is join irreducible if $a \neq x \vee y$ for $x, y \prec a$, alternatively
	- $a = x \vee y$ implies $a = x$ or $a = y$
- 0 is join irreducible

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•
$$
d_j = d_i \vee d_j
$$
 for $d_i \preceq d_j$

• Any $d_i \preceq d_i$ can be dropped to make the join irredundant

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	- $a = x \vee y$ implies $a = x$ or $a = y$
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- **•** If $b_1 \nless c$ and $b_2 \nless c$ (immediate preds) of *c* then $c = b_1 \vee b_2$
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- Elements immediately succeeding 0 are atoms (join irreducible)
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- Not unique for non-distributive lattice (diamond lattice)
- **•** For finite lattice $a = d_1 ∨ d_2 ∨ … ∨ d_n$, d_i are join irreducible

•
$$
d_j = d_i \vee d_j
$$
 for $d_i \preceq d_j$

- Any $d_i \preceq d_i$ can be dropped to make the join irredundant
- Unique (up to permutation) for distributive lattice

Epresentation (contd.)

Freducible sum representation
 \therefore \lor $c_m = d_1 \lor d_2 \lor \ldots \lor d_n$ **Boolean lattice representation (contd.)**

Unique irredundant irreducible sum representation

 \bullet Let *a* = *c*₁ ∨ *c*₂ ∨ ... ∨ *c_m* = *d*₁ ∨ *d*₂ ∨ ... ∨ *d_n*

 \rightarrow \equiv \rightarrow

Epresentation (contd.)

Freducible sum representation
 \therefore $\vee c_m = d_1 \vee d_2 \vee \ldots \vee d_n$
 $c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_n$ **Boolean lattice representation (contd.)**

Unique irredundant irreducible sum representation

- \bullet Let *a* = *c*₁ ∨ *c*₂ ∨ ... ∨ *c_m* = *d*₁ ∨ *d*₂ ∨ ... ∨ *d_n*
- $\mathsf{Now}, \, c_i, d_j \preceq c_1 \vee c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_m$
Unique irredundant irreducible sum representation

- \bullet Let $a = c_1 \vee c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_m$
- $\mathsf{Now}, \, c_i, d_j \preceq c_1 \vee c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_m$
- **epresentation (contd.)**

Freducible sum representation
 \therefore $\vee c_m = d_1 \vee d_2 \vee \ldots \vee d_n$
 $c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_n$
 $\langle \ldots \vee d_n \rangle = (c_i \wedge d_1) \vee (c_i \wedge d_2) \vee \ldots \vee (c_i \wedge d_n)$ ∴ $c_i = c_i \wedge (d_1 \vee d_2 \vee \ldots \vee d_n) = (c_i \wedge d_1) \vee (c_i \wedge d_2) \vee \ldots \vee (c_i \wedge d_n)$

Unique irredundant irreducible sum representation

- \bullet Let $a = c_1 \vee c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_m$
- $\mathsf{Now}, \, c_i, d_j \preceq c_1 \vee c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_m$
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Freducible sum representation
 \therefore $\vee c_m = d_1 \vee d_2 \vee \ldots \vee d_n$
 $c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_n$
 $\langle \ldots \vee d_n \rangle = (c_i \wedge d_1) \vee (c_i \wedge d_2) \vee \ldots \vee (c_i \wedge d_n)$

ducible, $\exists d_j | c_i = c_i \wedge d_j$, so ∴ $c_i = c_i \wedge (d_1 \vee d_2 \vee \ldots \vee d_n) = (c_i \wedge d_1) \vee (c_i \wedge d_2) \vee \ldots \vee (c_i \wedge d_n)$
- $\textsf{Since } c_i$ is join irreducible, $\exists d_j | c_i = c_i \land d_j,$ so that $c_i \preceq d_j$

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 $(1, 1)$ $(1, 1)$ $(1, 1)$ $(1, 1)$ $(1, 1)$ $(1, 1)$

Unique irredundant irreducible sum representation

- \bullet Let $a = c_1 \vee c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_m$
- $\mathsf{Now}, \, c_i, d_j \preceq c_1 \vee c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_m$
- **Epresentation (contd.)**

Freducible sum representation
 \therefore $\vee c_m = d_1 \vee d_2 \vee \ldots \vee d_n$
 $c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_n$
 $\wedge \ldots \vee d_n) = (c_i \wedge d_1) \vee (c_i \wedge d_2) \vee \ldots \vee (c_i \wedge d_n)$

ducible, $\exists d_j | c_i = c_i \wedge d_j$, so ∴ $c_i = c_i \wedge (d_1 \vee d_2 \vee \ldots \vee d_n) = (c_i \wedge d_1) \vee (c_i \wedge d_2) \vee \ldots \vee (c_i \wedge d_n)$
- $\textsf{Since } c_i$ is join irreducible, $\exists d_j | c_i = c_i \land d_j,$ so that $c_i \preceq d_j$
- But similar working, $d_j \preceq c_k$, so that $c_i \preceq d_j \preceq c_k$

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Unique irredundant irreducible sum representation

- \bullet Let $a = c_1 \vee c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_m$
- $\mathsf{Now}, \, c_i, d_j \preceq c_1 \vee c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_m$
- **Epresentation (contd.)**

Freducible sum representation
 \therefore $C_m = d_1 \vee d_2 \vee ... \vee d_n$
 $c_2 \vee ... \vee c_m = d_1 \vee d_2 \vee ... \vee d_n$
 $\wedge ... \vee d_n) = (c_i \wedge d_1) \vee (c_i \wedge d_2) \vee ... \vee (c_i \wedge d_n)$

ducible, $\exists d_j | c_i = c_i \wedge d_j$, so that $c_i \preceq d$ ∴ $c_i = c_i \wedge (d_1 \vee d_2 \vee \ldots \vee d_n) = (c_i \wedge d_1) \vee (c_i \wedge d_2) \vee \ldots \vee (c_i \wedge d_n)$
- $\textsf{Since } c_i$ is join irreducible, $\exists d_j | c_i = c_i \land d_j,$ so that $c_i \preceq d_j$
- But similar working, $d_j \preceq c_k$, so that $c_i \preceq d_j \preceq c_k$
- This requires $c_i = c_k$, since these are irredundant

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Unique irredundant irreducible sum representation

- \bullet Let $a = c_1 \vee c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_m$
- $\mathsf{Now}, \, c_i, d_j \preceq c_1 \vee c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_m$
- **Epresentation (contd.)**

Freducible sum representation
 \therefore $\vee c_m = d_1 \vee d_2 \vee \ldots \vee d_n$
 $c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_n$
 $\wedge \ldots \vee d_n) = (c_i \wedge d_1) \vee (c_i \wedge d_2) \vee \ldots \vee (c_i \wedge d_n)$

ducible, $\exists d_j | c_i = c_i \wedge d_j$, so ∴ $c_i = c_i \wedge (d_1 \vee d_2 \vee \ldots \vee d_n) = (c_i \wedge d_1) \vee (c_i \wedge d_2) \vee \ldots \vee (c_i \wedge d_n)$
- $\textsf{Since } c_i$ is join irreducible, $\exists d_j | c_i = c_i \land d_j,$ so that $c_i \preceq d_j$
- But similar working, $d_j \preceq c_k$, so that $c_i \preceq d_j \preceq c_k$
- This requires $c_i = c_k$, since these are irredundant

• Thus,
$$
c_i \preceq d_j
$$
 and $d_j \preceq c_i$, $c_i = d_j$,

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Unique irredundant irreducible sum representation

- \bullet Let $a = c_1 \vee c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_n$
- $\mathsf{Now}, \, c_i, d_j \preceq c_1 \vee c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_m$
- **Epresentation (contd.)**

Freducible sum representation
 \therefore $C_m = d_1 \vee d_2 \vee ... \vee d_n$
 $c_2 \vee ... \vee c_m = d_1 \vee d_2 \vee ... \vee d_n$
 $\langle ... \vee d_n \rangle = (c_i \wedge d_1) \vee (c_i \wedge d_2) \vee ... \vee (c_i \wedge d_n)$

ducible, $\exists d_j | c_i = c_i \wedge d_j$, so that $c_i \preceq d$ ∴ $c_i = c_i \wedge (d_1 \vee d_2 \vee \ldots \vee d_n) = (c_i \wedge d_1) \vee (c_i \wedge d_2) \vee \ldots \vee (c_i \wedge d_n)$
- $\textsf{Since } c_i$ is join irreducible, $\exists d_j | c_i = c_i \land d_j,$ so that $c_i \preceq d_j$
- But similar working, $d_j \preceq c_k$, so that $c_i \preceq d_j \preceq c_k$
- This requires $c_i = c_k$, since these are irredundant

• Thus,
$$
c_i \preceq d_j
$$
 and $d_j \preceq c_i$, $c_i = d_j$,

This way, all the *ci*s may to paired off with the *dj*s,

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 $\left\{ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right.$

Unique irredundant irreducible sum representation

- \bullet Let $a = c_1 \vee c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_n$
- $\mathsf{Now}, \, c_i, d_j \preceq c_1 \vee c_2 \vee \ldots \vee c_m = d_1 \vee d_2 \vee \ldots \vee d_m$
- ∴ $c_i = c_i \wedge (d_1 \vee d_2 \vee \ldots \vee d_n) = (c_i \wedge d_1) \vee (c_i \wedge d_2) \vee \ldots \vee (c_i \wedge d_n)$
- $\textsf{Since } c_i$ is join irreducible, $\exists d_j | c_i = c_i \land d_j,$ so that $c_i \preceq d_j$
- But similar working, $d_j \preceq c_k$, so that $c_i \preceq d_j \preceq c_k$
- This requires $c_i = c_k$, since these are irredundant

• Thus,
$$
c_i \preceq d_j
$$
 and $d_j \preceq c_i$, $c_i = d_j$,

Epresentation (contd.)

Freducible sum representation
 \therefore $C_m = d_1 \vee d_2 \vee ... \vee d_n$
 $c_2 \vee ... \vee c_m = d_1 \vee d_2 \vee ... \vee d_n$
 $\langle ... \vee d_n \rangle = (c_i \wedge d_1) \vee (c_i \wedge d_2) \vee ... \vee (c_i \wedge d_n)$

ducible, $\exists d_j | c_i = c_i \wedge d_j$, so that $c_i \preceq d$ This way, all the *ci*s may to paired off with the *dj*s, making the representation unique (up to permutation)

 $\left\{ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right.$

Boolean lattice structure (contd.)

- Let *z* be the complement of *a* in a lattice as shown
- Suppose *a* has *b* as a unique predecessor
- So, *a* ∨ *z* = 1 and *a* ∧ *z* = 0
- **Example 3 [Boolean lattice structure](#page-79-0)**
 control of a in a lattice as shown

a unique predecessor
 $\begin{vmatrix} x & 2 \\ y & 2 \end{vmatrix}$
 $\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix}$
 $\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix}$

sor of a

ement of z Now, *b* ∨ *z* = *a* ∨ *z* = 1 and *b* ∧ *z* = *a* ∧ *z* = 0 as *b* is the immediate predecessor of *a*
- So, *b* is also a complement of *z*

 $z \neq 1$ $b \neq 0$

a

Boolean lattice structure (contd.)

- Let *z* be the complement of *a* in a lattice as shown
- Suppose *a* has *b* as a unique predecessor
- So, *a* ∨ *z* = 1 and *a* ∧ *z* = 0
- Now, *b* ∨ *z* = *a* ∨ *z* = 1 and *b* ∧ *z* = *a* ∧ *z* = 0 as *b* is the immediate predecessor of *a*
- So, *b* is also a complement of *z*

Join irreducible elements in a Boolean lattice

- A lattice with having a non-zero join irreducible element will not have unique complements
- **•** In a Boolean lattice all non-zero join irreducible elements are atoms

a

 $b \neq 0$

 $z \neq 1$

- Atom of a Boolean lattice: Non-trivial minimal element of $A \setminus \{0\}$
- $|A| = 2ⁿ$ for some *n* for a Boolean lattice
- kOflm^**[Discrete structures](#page-81-0) [Boolean lattice structure](#page-81-0)** • Its structure is that of the power set of the atomic elements

- Atom of a Boolean lattice: Non-trivial minimal element of $A \setminus \{0\}$
- $|A| = 2ⁿ$ for some *n* for a Boolean lattice
- • Its structure is that of the power set of the atomic elements
- **ation of Boolean lattice** structure
 ation of Boolean lattice: Non-trivial

minimal element of $A \setminus \{0\}$
 e $|A| = 2^n$ for some *n* for a Boolean lattice
 e Its structure is that of the power set of the

atomic el • Non-trivial atomic elements are present for $|A| > 1$ directly above level 0, let those be $S = \{a_1, \ldots, a_n\}$, akin to $\{a_1\}$, $\{a_2\}$, ... $\{a_n\}$

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- Join of pairs of elements Y_1 , Y_2 at level *i* ($n > i > 1$) st $|Y_1 - Y_2| = |Y_2 - Y_1| = 1$ at level $i + 1$ is $Y = Y_1 \cup Y_2$

- Atom of a Boolean lattice: Non-trivial minimal element of $A \setminus \{0\}$
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- Its structure is that of the power set of the atomic elements
- **ation of Boolean lattice:** Non-trivial

minimal element of $A \setminus \{0\}$

 $|A| = 2^n$ for some *n* for a Boolean lattice

 Its structure is that of the power set of the

atomic elements

elements are present for $|A| > 1$ • Non-trivial atomic elements are present for $|A| > 1$ directly above level 0, let those be $S = \{a_1, \ldots, a_n\}$, akin to $\{a_1\}$, $\{a_2\}$, ... $\{a_n\}$
- Join of pairs of elements Y_1 , Y_2 at level *i* $(n > i > 1)$ st $|Y_1 - Y_2| = |Y_2 - Y_1| = 1$ at level $i + 1$ is $Y = Y_1 \cup Y_2$
- Meet of pairs of elements X_1 , X_2 at level *i* $(n > i > 1)$ st $|X_1 - X_2| = |X_2 - X_1| = 1$ at level $i - 1$ is $Y = Y_1 \cap Y_2$

 $\left\{ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right.$

- Atom of a Boolean lattice: Non-trivial minimal element of $A \setminus \{0\}$
- $|A| = 2ⁿ$ for some *n* for a Boolean lattice
- Its structure is that of the power set of the atomic elements
- **ation of Boolean lattice:** Non-trivial
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minimal element of $A \setminus \{0\}$
 e $|A| = 2^n$ for some *n* for a Boolean lattice
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atomic • Non-trivial atomic elements are present for $|A| > 1$ directly above level 0, let those be $S = \{a_1, \ldots, a_n\}$, akin to $\{a_1\}$, $\{a_2\}$, ... $\{a_n\}$
- Join of pairs of elements Y_1 , Y_2 at level *i* $(n > i > 1)$ st $|Y_1 - Y_2| = |Y_2 - Y_1| = 1$ at level $i + 1$ is $Y = Y_1 \cup Y_2$
- Meet of pairs of elements X_1 , X_2 at level *i* $(n > i > 1)$ st
	- $|X_1 X_2| = |X_2 X_1| = 1$ at level $i 1$ is $Y = Y_1 \cap Y_2$

There will be $\binom{n}{i}$ $\binom{n}{i}$ such sets in level *i*, totali[ng](#page-84-0) [to](#page-86-0) $\sum_{i=0}^n \binom{n}{i}$ $\sum_{i=0}^n \binom{n}{i}$ $\sum_{i=0}^n \binom{n}{i}$ $\binom{n}{i} = 2^n$ **^N^I ^D ^A^I ^N^S^T^ITUT^E ^O^F ^T^E^C^HNOLOG^Y ^K^H^A R ^G^P^UR** \sim --yog, km
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Boolean algebra from Boolean lattice

- **from Boolean lattice**

from Boolean lattice

from $x \mapsto x$, $0, 1$ consider the algebraic system
 $x \in \mathbb{R}$
 $x = 0$

the Huntington's postulates for a [Boolean algebra](#page-86-0)
 $y + x$
 $y + x$
 $y + x$
 $y + x$
 $y x$
 $y x$
 $y x$
 y • For the Boolean lattice $\langle A, \prec, 0, 1 \rangle$ consider the algebraic system $\langle A, +, \cdot, -, 0, 1 \rangle$ where $\vee \mapsto +, \wedge \mapsto \cdot$ and $\forall x \in A$, $\overline{x} \mapsto z|x+z=1, x \cdot z=0$
- **•** This system satisfies the Huntington's postulates for a Boolean algebra

B1: Commutative Laws

$$
\begin{array}{c}\n\bullet \quad x + y = y + x \\
\bullet \quad x \cdot y = y \cdot x\n\end{array}
$$

B2: Distributive Laws

$$
\begin{array}{c}\n\bullet \quad x \cdot (y+z) = x \cdot y + x \cdot z \\
\bullet \quad x + (y \cdot z) = (x+y) \cdot (x+z)\n\end{array}
$$

B3: Identity Laws

$$
x + 0 = x = 0 + x
$$

2 $x \cdot 1 = x = 1 \cdot x$

B4: Complementation Laws

$$
\begin{array}{c}\n\bullet \quad x + \bar{x} = 1 = \bar{x} + x \\
\bullet \quad x \cdot \bar{x} = 0 = \bar{x} \cdot x\n\end{array}
$$

Additional Boolean algebra properties

- These properties carry over from the Boolean lattice
- May be proven independently from the Huntington's postulates

Idempotence:

$$
x + x = x
$$

2 $x \cdot x = x$

Absorption:

\n- **①**
$$
x + xy = x
$$
\n- **②** $x \cdot (x + y) = x$
\n- **③** $x + \overline{x}y = x + y$
\n- **④** $x \cdot (\overline{x} + y) = xy$
\n

Axiomatic proof

$$
x + x = (x + x) \cdot 1
$$

$$
\begin{array}{l}\n\bullet = (x + x) \cdot (x + \bar{x}) \\
\bullet = x + (x \cdot \bar{x}) \\
\bullet = x + 0 = x\n\end{array}
$$

Axiomatic proof

kOflm^**[Discrete structures](#page-87-0) [Additional BA properties](#page-87-0)** *x* + *xy* = (*x* · 1) + *xy* = *x*(1 + *y*) = *x*(*y* + 1) = *x* · 1 = *x*

4 m + 4 m

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Axiomatic proof

Boolean algebra (contd.)

Boundedness/annihilation:

Associativity:

$$
\begin{array}{c}\n\bullet \ (x+y)+z=x+(y+z) \\
\bullet \ (x \cdot y) \cdot z=x \cdot (y \cdot z)\n\end{array}
$$

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Boolean algebra (contd.)

Axiomatic proof of associativity of Boolean +

• Let
$$
x = a + (b + c)
$$
 and $y = (a + b) + c$

•
$$
ax = aa + a(b + c) = a + a(b + c) = a
$$

•
$$
bx = ba + b(b + c) = ba + (bb + bc) = ba + (b + bc) = ba + b = b
$$

 \bullet Similarly, $cx = c$ and $ay = a$, $by = b$ and $cy = c$

kOflm^**[Discrete structures](#page-89-0) [Additional BA properties](#page-89-0)** *yx* = ((*a*+*b*)+*c*)*x* = (*a*+*b*)*x*+*cx* = (*ax*+*bx*)+*cx* = (*a*+*b*)+*c* = *y*

•
$$
xy = (a+(b+c))y = ay+(b+c)y = ay+(by+cy) = a+(b+c) = x
$$

• Thus,
$$
x = xy = yx = y
$$

4 m + 4 m

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Additional Boolean algebra properties (contd.)

Uniqueness of Complement:

If
$$
(a+x) = 1
$$
 and $(a \cdot x) = 0$, then $x = \overline{a}$.

Involution:

$$
\overline{(\overline{a})}=a
$$

Complements of extreme elements:

Axiomatic proof

$$
\bullet~\overline{\underline{0}}=1
$$

 \bullet 1 = 0

- \bullet 1 + 0 = 1 [identity]
- \bullet 1 \cdot 0 = 0 [boundedness]
- ∴ 0 is the complement of 1

DeMorgan's laws:

$$
\begin{array}{c}\n\bullet \overline{(x+y)} = \overline{x} \cdot \overline{y} \\
\bullet \overline{(x \cdot y)} = \overline{x} + \overline{y}\n\end{array}
$$

• Let $a \prec b$ if $a \cdot b = a$ or $a + b = b$ then $\cdot \mapsto \wedge$ and $+ \mapsto \vee$

Ram algebra properties (contd.)
 Lement:
 $= 1$ and $(a \cdot x) = 0$, then $x = \overline{a}$.
 eme elements:

Axiomatic proof
 $\bullet 1 + 0 = 1$ [identity]
 $\bullet 1 \cdot 0 = 0$ [boundedness]
 $\therefore 0$ is the complement of 1
 \overline{y}) $= \overline$ • Properties from axiomatic proofs allow Boolean algebras to be expressed as Boolean lattices – they are [eq](#page-89-0)[uiv](#page-90-0)[a](#page-89-0)[len](#page-90-0)[t](#page-86-0)

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