An Introduction to Ramsey Theory
OUTLINE

• History
• Pigeonhole
• Ramsey Number
• Other Ramsey Theory
• Thinking about Ramsey
History

• Frank Plumpton Ramsey 1903–1930
• British mathematician
• Ramsey is a **BRANCH** of theory
• Ramsey theory ask: "how many elements of some structure must there be to guarantee that a particular property will hold?"
Pigeonhole

- Pigeonhole is a **simple** theory
  - m objects divide into n classes
  - at least \([m/n]\) objects appears
- Application can be **subtle**
- Pigeonhole and Ramsey are closely linked
  - Some Ramsey can be proved by Pigeonhole
  - They both satisfy “how many elements can guarantee a property”
• Ramsey numbers part of mathematical field of graph theory

• $k_m$ is defined as a graph containing $m$ nodes and all possible line between the nodes

• Ramsey functions notated as $K(r, b)=n$
  – $K$ is Ramsey function
  – $r, b$ are independent variables
  – $n$ is result of Ramsey function; called Ramsey number

• Ramsey function gives smallest graph size that when colored in any pattern of only two colors, will not contain sub-graphs of size $r$ or $b$ (i.e. does not contain a $k_r$ or $k_b$)
• The Ramsey number $R(m,n)$ gives the solution to the party problem, which asks the minimum number of guests $R(m,n)$ that must be invited so that at least $m$ will know each other or at least $n$ will not know each other.

• $R(X,1)=R(1,X)=1$

• $R(2,X)=X$

• $R(3,3)=??$
Ramsey Number

Example: $R(3,3) = 6$

- Is the 3rd Ramsey number 5?

  ![No](image)

  Possible to create a mapping without a $k_3$ in a 5-node graph (a $k_5$)

- Is the 3rd Ramsey number 6?

  ![Yes](image)

  Not possible to create a coloring without a $k_3$ in a 6-node graph (a $k_6$) and this $R(3,3) = 6$
Ramsay Theorem

• In the language of graph theory, the Ramsey number is the minimum number of vertices $v=R(m,n)$, such that all undirected simple graphs of order $v$ contain a clique of order $m$ or an independent set of order $n$.

• Ramsey theorem states that such a number exists for all $m$ and $n$. 
Existence Proof of $R(r,s)$

- for the 2-colour case, by induction on $r + s$
- Existance proof by proving a explicit bound
- base: for all $n$, $R(n, 1) = R(1, n) = 1$
- By the inductive hypothesis $R(r - 1, s)$ and $R(r, s - 1)$ exist.
- **Claim:** $R(r, s) \leq R(r - 1, s) + R(r, s - 1)$
Existence Proof of $R(r, s)$

1. Consider a complete graph on $R(r - 1, s) + R(r, s - 1)$ vertices.
2. Pick a vertex $v$ from the graph, and partition the remaining vertices into two sets $M$ and $N$, such that for every vertex $w$, $w$ is in $M$ if $(v, w)$ is blue, and $w$ is in $N$ if $(v, w)$ is red.
3. Because the graph has $R(r - 1, s) + R(r, s - 1) = |M| + |N| + 1$ vertices, it follows that either
   $$|M| \geq R(r - 1, s) \text{ or } |N| \geq R(r, s - 1)$$
4. In the former case, if $M$ has a red $K_s$ then so does the original graph and we are finished.
5. Otherwise $M$ has a blue $K_{r-1}$ and so $M \cup \{v\}$ has blue $K_r$ by definition of $M$. The latter case is analogous.
6. $R(4, 3) =$ ??

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Small Ramsey Numbers

- $R(4,3) \leq R(3,3) + R(4,2) \\ \leq 6 + 4 = 10$
- $R(4,3) = 9$

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<th>M</th>
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<td>[43, 49]</td>
<td>Exoo 1989b, McKay and Radziszowski 1995</td>
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<td>[102, 165]</td>
<td>Kalbfleisch 1965, Mackey 1994</td>
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<td>19</td>
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<td>[17885, 9075135299]</td>
<td>Luo et al. 2002</td>
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A generalized Ramsey number

- A generalized Ramsey number is written
  \[ r = R(M_1, M_2, \ldots, M_k; n) \]
- It is the smallest integer \( r \) such that, no matter how each \( n \)-element subset of an \( r \)-element sets is colored with \( k \) colors, there exists an \( i \) such that there is a subset of size \( M_i \), all of whose \( n \)-element subsets are color \( i \).
A generalized Ramsey number

- \( R(M_1, M_2, \ldots, M_k; n) \)
- when \( n > 2 \), little is known.
  - \( R(4,4,3)=13 \)

- When \( k > 2 \), little is known.
  - \( R(3,3,3)=14 \)

- Ramsey number tell us that \( R(m_1, m_2, \ldots, m_k; n) \) always exist!
Other Ramsey Theory

- Graph Ramsey Number
- Ramsey Polygon Number
- Ramsey of Bipartite graph
- .......
Graph Ramsey Number

• Given simple graphs $G_1, \ldots, G_k$, the graph Ramsey number $R(G_1, \ldots, G_k)$ is the smallest integer $n$ such that every $k$-coloring of $E(K_n)$ contains a copy of $G_i$ in color $i$ for some $i$. 
For any integers $m$ and $c$, and any integers $n_1, \ldots, n_c$, there is an integer $R(n_1, \ldots, n_c; c, m)$ such that if the hyperedges of a complete $m$-hypergraph of order $R(n_1, \ldots, n_c; c, m)$ are coloured with $c$ different colours, then for some $i$ between 1 and $c$, the hypergraph must contain a complete sub-$m$-hypergraph of order $n_i$ whose hyperedges are all colour $i$. 

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Ramsey Theory Applications

• Number Theory: Schur’s theorem
  
  *if* $N$ *is partitioned into a finite number of classes, at least one partition class contains a solution to the equation* $x + y = z$.

• Computational geometry: Erdos-Szekeres theorem

  $2^{n-2} + 1 \leq g(n) \leq \binom{2n-4}{n-2}$
  
  where $g(n)$ denotes the smallest number such that any set of at least $g(n)$ points in general position in the plane contains $n$ points in convex position. The Erdos-Szekeres theorem is the consequence of the finite Ramsey theorem.
Final Thoughts

• Results in Ramsey theory typically have two primary characteristics:
  – **non-constructive**: exist but non-constructive
    • This is same for pigeonhole
  – **Grow exponentially**: results requires these objects to be enormously large.
    • That’s why we still know small ramsey number
    • Computer is useless here!
The reason behind such Ramsey-type results is that: “The largest partition class always contains the desired substructure”.

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Questions
Thank you