Ramsey numbers for complete bipartite and 3-uniform tripartite subgraphs

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Ramsey Number $R(a, b)$

$R(a, b)$ is the minimum number n such that any bicoloring of the edges of the n-vertex complete undirected graph K_n would contain a monochromatic K_a or a monochromatic K_b.

$$R(1,1) = 1, R(1,b) = 1$$
$$R(2,2) = 2, R(2,b) = b$$
$$R(3,3) = 6 \; R(4,4) = 18 \; R(4,5) = 25$$

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$$R(6,6) = 102 - 165$$
$$R(10,10) = 798 - 23556$$

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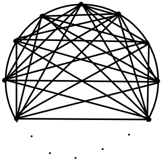

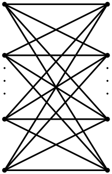

Traditional Ramsey searches for complete structures (like K_a or K_b), but what happens if we try to find complete bipartite structures?
Solving this basic question is the area of my research.

Definition of $R'(a, b)$

$R'(a, b)$ is the minimum number n such that any bicoloring of the edges of the n-vertex complete undirected graph K_n would contain a monochromatic $K_{a,b}$.

In other words, $R'(a, b)$ is the minimum number n so that any n-vertex simple undirected graph G or its complement G' must contain the complete bipartite graph $K_{a,b}$.

$R'(a, b)$ is the minimum number n such that any bicoloring of the edges of the n-vertex complete undirected graph K_n would contain a monochromatic $K_{a,b}$.

$R'(1, 1) = ?$

any bicoloring of the edges of the $R'(1, 1)$-vertex complete undirected graph would contain a monochromatic $K_{1,1}$.

Figure 1 : monochromatic $K_{1,1}$ in bicoloring using red and blue.

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Figure 1 : monochromatic $K_{1,1}$ in bicoloring using red and blue.

$R'(1, 1) = 2$

$R'(a, b)$ is the minimum number n such that any bicoloring of the edges of the n-vertex complete undirected graph K_n would contain a monochromatic $K_{a,b}$.

$R'(1, 2) = ?$

any bicoloring of the edges of the $R'(1, 2)$-vertex complete undirected graph would contain a monochromatic $K_{1,2}$.

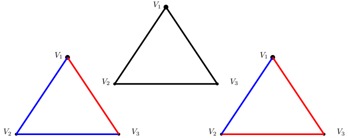

Figure 2 : monochromatic $K_{1,2}$ in bicoloring using red and blue.

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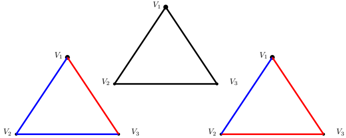


Figure 2 : monochromatic $K_{1,2}$ in bicoloring using red and blue.

$R'(1,2) = 3$

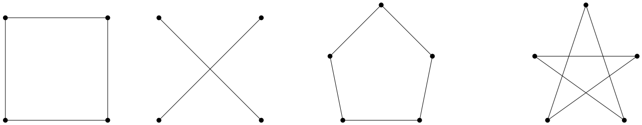
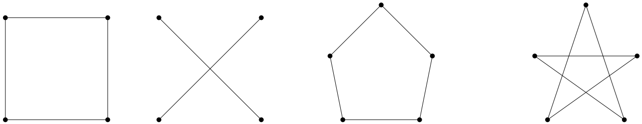


Figure 3 : $K_{2,2}$ free graphs with $n = 4$ and $n = 5$ vertices.

$R'(1,3) \geq 6$, observe that we need at least 4 vertices and neither a 4-cycle nor it complement has a $K_{1,3}$. Further, observe that neither a 5-cycle in K_5, nor its complement (also a 5-cycle) has a $K_{1,3}$.

- Non-constructive



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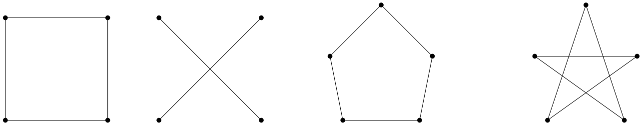
- Non-constructive
- Monotonically increasing



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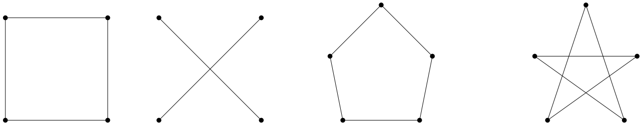
- Non-constructive
- Monotonically increasing
- Grows exponentially



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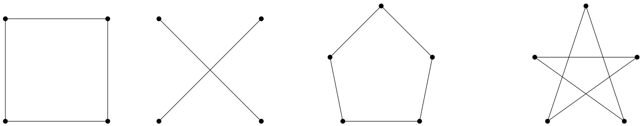

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- Monotonically increasing
- Grows exponentially

$R'(7,7) \leq 125500$
$R'(8,8) \leq 7456621$

Results

1. $R'(1, b) = 2b$, if b is odd.
   $R'(1, b) = 2b - 1$, if b is even.
2. $R'(2, b) > 2b + 1$, for all integers $b \geq 2$.
3. $R'(a, b) > \dfrac{(2\pi)^{\left(\frac{1}{a+b}\right)} * a^{\left(\frac{a+\frac{1}{2}}{a+b}\right)} * b^{\left(\frac{b+\frac{1}{2}}{a+b}\right)}}{e} * 2^{\left(\frac{ab-1}{a+b}\right)}$
4. If $e * 2^{1-ab} * \left(ab\binom{n-2}{a+b-2}\binom{a+b-2}{b-1} + 1\right) \leq 1,\ R'(a, b) > n$
5. For all $n \in N$ and $0 < p < 1$, if
   $\binom{n}{a}\binom{n-a}{b}p^{ab} + \binom{n}{c}\binom{n-c}{d}(1 - p)^{cd} < 1$, then
   $R'(a, b, c, d) > n$.

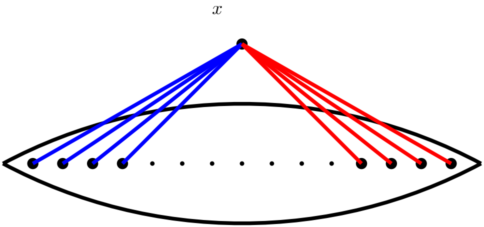
Results Continued..

6. $R'(p,q) \leq R(p+q, p+q)$.
7. $R'(a,b) \leq 2^a * R'(a-1,b)$, $a < b$.
8. if $\frac{n(n-1)}{2} \geq 2 * \frac{a}{2} * \sqrt[a]{\frac{b-1}{a!}} * n^{2-\frac{1}{a}} + 1$, $R'(a,b) < n$.
9. $R'(a,b,c) >$
$\frac{(2\pi)^{\left(\frac{3}{2(a+b+c)}\right)} * a^{\left(\frac{a+\frac{1}{2}}{a+b+c}\right)} * b^{\left(\frac{b+\frac{1}{2}}{a+b+c}\right)} * c^{\left(\frac{c+\frac{1}{2}}{a+b+c}\right)}}{e} * 2^{\left(\frac{abc-1}{a+b+c}\right)}$.
10. If $e * 2^{1-abc} * \left(abc\binom{n-3}{a+b+c-3}\binom{a+b+c-3}{b-1}\binom{a+c-2}{c-1} + 1\right) \leq 1$, $R'(a,b,c) > n$.
11. (Conjecture)$R'(1,1,b) = b+2$.

Theorem

$2b - 1 \leq R'(1, b) \leq 2b.$

$R'(1, b) \leq 2b$: $n = 2b$ vertices:
for any vertex x, there are exactly $2b - 1$ possible neighbours, so by pigeon hole principle, x must contain b neighbours in atleast one of G or G'. Those b neighbours combined with x forms the $K_{1,b}$.



$2b - 1$ possible neighbours

$2b - 1 \leq R'(1, b) \leq 2b$.

$R'(1, b) \geq 2b - 1$: $n = 2b - 2$(i.e $< 2b - 1$) vertices:
To show that $R'(1, b) \geq 2b - 1$, we need to give a general construction with $2b - 2$ vertices graphs G and G' free from $K_{1,b}$. So our construction would generate a graph G that is $(b - 1)$-regular(that will be obviously free from $K_{1,b}$), such that the number of possible neighbours for any vertex in G' cannot exceed $b - 1$.

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Construction of G: If $b - 1 = 2m$ is even, put all the vertices around a circle, and join each to its m nearest neighbors on either side.

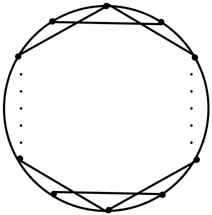

Figure 4 : $b - 1(= 2m)$ is even, $m = 2$ in here

$R'(1, b) \geq 2b - 1$

Construction of G: If $b - 1 = 2m$ is even, put all the vertices around a circle, and join each to its m nearest neighbors on either side.
If $b - 1 = 2m + 1$ is odd (and as $n = 2b - 2$ is even), put the vertices on a circle, join each to its m nearest neighbors on each side, and also to the vertex directly opposite.

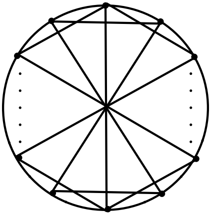

Figure 5 : $b - 1(= 2m + 1)$ is odd, $m = 2$ here

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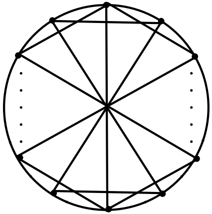

Figure 5 : $b - 1(= 2m + 1)$ is odd, $m = 2$ here

This will result in a $(b - 1)$-regular graph G such that G and its complement G' are free from $K_{1,b}$.

Theorem

$R'(1, b) = 2b$, *if b is odd.*
$R'(1, b) = 2b - 1$, *if b is even.*

$R'(2,2) > 5$

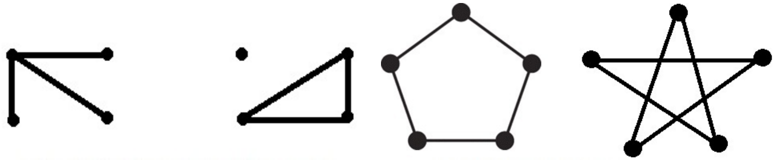

Figure 6 : G and G' with $n = 4$ and $n = 5$ free from a $K_{2,2}$

$R'(2,2) = 6$.

$R'(2,3) > 7$

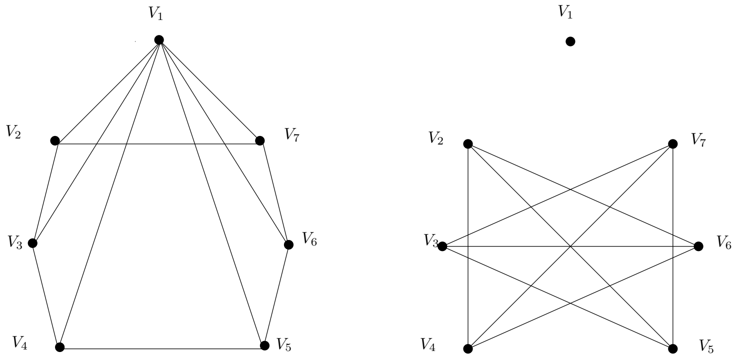

Figure 7 : G and G' with $n = 7$ without a $K_{2,3}$

Theorem

$R'(2, b) > 2b + 1$, *for all integers* $b \geq 2$.

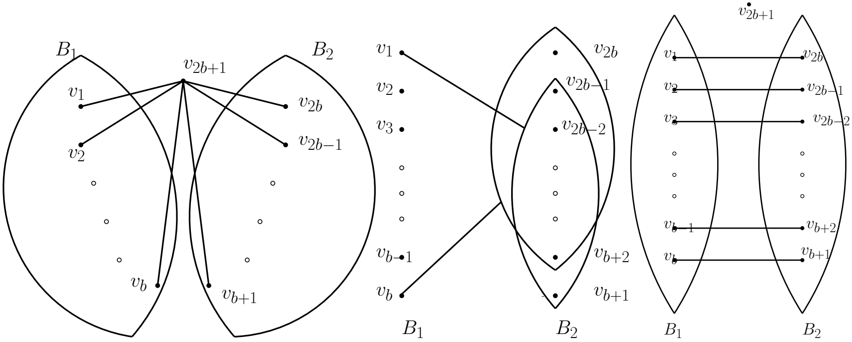

Figure 8 : Construction of G(left two): generation of B_1, B_2 and addition of edges. Resulting G'(rightmost): In G', B_1 and B_2 become K_b, and only edges between B_1 and B_2 is a matching.

Theorem

$R'(3,3) > 11$.

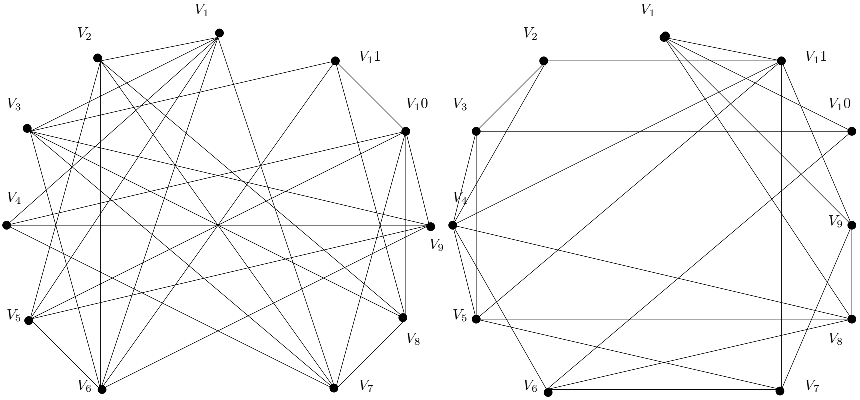

Figure 9 : G and G' with $n = 11$ without a $K_{3,3}$

Probabilistic lower bounds for $R'(a, b)$

we want some (ideally as large as possible) n so that we can somehow colour the edges of K_n using two colors (say red and blue) in such a way that we get neither a red K_a or a blue K_b.

non-constructive, but shows such examples exist!

Earlier works..

The best known lower bound on $R'(a, b)$ due to Chung and Graham [4] is

$$R'(a, b) > \left(2\pi\sqrt{ab}\right)^{\left(\frac{1}{a+b}\right)} * \left(\frac{a+b}{e^2}\right) * 2^{\frac{ab-1}{a+b}} \tag{1}$$

Probablistic Lower Bound

Theorem

$$R'(a,b) > \frac{(2\pi)^{\left(\frac{1}{a+b}\right)} * a^{\left(\frac{a+\frac{1}{2}}{a+b}\right)} * b^{\left(\frac{b+\frac{1}{2}}{a+b}\right)}}{e} * 2^{\left(\frac{ab-1}{a+b}\right)}$$

Proof: Let n be the number of vertices of graph G. Then the total number of distinct $K_{a,b}$ possible is

$$\binom{n}{a} * \binom{n-a}{b}$$

Probablistic Lower Bound

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Proof: Let n be the number of vertices of graph G. Then the total number of distinct $K_{a,b}$ possible is

$$\binom{n}{a} * \binom{n-a}{b}$$

Each $K_{a,b}$ has exactly ab edges. Each edge can be either of color 1 or color 2 with equal probability. So probability of a particular $K_{a,b}$ of color 1 is $\left(\frac{1}{2}\right)^{ab}$. So probability that a particular $K_{a,b}$ of either color 1 or color 2 exists is

$$2 * \left(\frac{1}{2}\right)^{ab} = 2^{1-ab} \quad (2)$$

Probablistic Lower Bound

So probalility p of any monochromatic $K_{a,b} =$

$$\binom{n}{a} * \binom{n-a}{b} * 2^{1-ab} \tag{3}$$

Our objective is to choose as large n as possible with $p < 1$. So choosing

$$n = \frac{2\pi^{\left(\frac{1}{a+b}\right)} * a^{\left(\frac{a+\frac{1}{2}}{a+b}\right)} * b^{\left(\frac{b+\frac{1}{2}}{a+b}\right)}}{e} * 2^{\left(\frac{ab-1}{a+b}\right)}$$, we get $p < 1$.

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This guarantees the existence of an n-vertex graph for which some edge bicoloring would not result in any monochromatic $K_{a,b}$. □

A lower bound for $R'(a, b)$ using Lovász' local lemma

Objective: existence of a monochromatic $K_{a,b}$ in any bicoloring of the edges of K_n.

Since the same edge may be present in many distinct $K_{a,b}$'s, the colouring of any particular edge may effect the monochromaticity in many $K_{a,b}$'s. This gives the motivation of use of Lovász' local lemma (see [9]) in this context.

Theorem (Lovász' local lemma Corrolary)

If every event E_i, $1 \leq i \leq m$ is dependent on at most d other events and $Pr\left[E_i\right] \leq p$, and if $ep(d+1) \leq 1$, then $Pr\left[\bigcap_{i=1}^{n} \overline{E_i}\right] > 0$.

Improved bound using LLL

Theorem

If $e * 2^{1-ab} * \left(ab\binom{n-2}{a+b-2}\binom{a+b-2}{b-1} + 1\right) \leq 1$, $R'(a,b) > n$

Proof: Let S be the set of edges of an arbitrary $K_{a,b}$, and let E_S be the event that all edges in this $K_{a,b}$ are coloured monochromatically.

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Proof: Let S be the set of edges of an arbitrary $K_{a,b}$, and let E_S be the event that all edges in this $K_{a,b}$ are coloured monochromatically. For each such S, the probability of E_S is $P\left(E_S\right) = 2^{1-ab}$.

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We enumerate the sets of edges of all possible $K_{a,b}$'s as S_1,S_2,...,S_m, where $m = \binom{n}{a}\binom{n-a}{b}$.

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We enumerate the sets of edges of all possible $K_{a,b}$'s as S_1,S_2,...,S_m, where $m = \binom{n}{a}\binom{n-a}{b}$.

Each event E_{S_i} is mutually independent of all the events E_{S_j} from the set

$$\{E_{S_j} : |S_i \cap S_j| = 0\} \tag{4}$$

since for any such S_j , S_i and S_j share no edges.

Improved bound using LLL Cont..

Theorem

If $e * 2^{1-ab} * \left(ab\binom{n-2}{a+b-2}\binom{a+b-2}{b-1} + 1 \right) \leq 1,\ R'(a,b) > n$

For each E_{S_i}, the number of events outside this set satisfies the inequality
$|\{E_{S_j} : |S_i \cap S_j| \geq 1\}| \leq ab\binom{n-2}{a+b-2}\binom{a+b-2}{b-1}$
as every S_j in this set shares at least one edge with S_i, and therefore such an S_j shares at least two vertices with S_i.

Improved bound using LLL Cont..

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as every S_j in this set shares at least one edge with S_i, and therefore such an S_j shares at least two vertices with S_i.
We can choose the rest of the $a + b - 2$ vertices of S_j from the remaining $n - 2$ vertices of K_n, out of which we can choose $b - 1$ for one partite of S_j, and the remaining $a - 1$ to form the second partite of S_j, yielding a $K_{a,b}$ that shares at least one edge with S_i.

Improved bound using LLL Cont..

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We apply Corollary 6 to the set of events $E_{S_1}, E_{S_2}, \ldots, E_{S_m}$, with

$$p = 2^{1-ab}\ ,\ d = ab\binom{n-2}{a+b-2}\binom{a+b-2}{b-1}\ , \qquad (5)$$

Improved bound using LLL Cont..

Theorem

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We apply Corollary 6 to the set of events $E_{S_1}, E_{S_2}, \ldots, E_{S_m}$, with $p = 2^{1-ab}$, $d = ab\binom{n-2}{a+b-2}\binom{a+b-2}{b-1}$,

$$e * 2^{1-ab} * \left(ab\binom{n}{a+b-2}\binom{a+b-2}{b-1} + 1\right) \leq 1 => Pr\left[\bigcap_{i=1}^{m} \overline{E}_{S_i}\right] > 0 \tag{6}$$

Improved bound using LLL Cont..

Theorem

If $e * 2^{1-ab} * \left(ab\binom{n-2}{a+b-2}\binom{a+b-2}{b-1} + 1\right) \leq 1,\ R'(a,b) > n$

We apply Corollary 6 to the set of events $E_{S_1}, E_{S_2}, \ldots, E_{S_m}$, with $p = 2^{1-ab}$, $d = ab\binom{n-2}{a+b-2}\binom{a+b-2}{b-1}$,

$$e * 2^{1-ab} * \left(ab\binom{n}{a+b-2}\binom{a+b-2}{b-1} + 1\right) \leq 1 => Pr\left[\bigcap_{i=1}^{m} \overline{E}_{S_i}\right] > 0 \quad (6)$$

This non-zero probability (of none of the events E_{S_i} occuring, for $1 \leq i \leq m$) implies the existence of some bicolouring of the edges of K_n with no monochromatic $K_{a,b}$, thereby establishing the theorem. □

Table 1 : Lower bounds for $R'(a, b)$ from Inequality 1(left), Theorem 5 (middle) and Theorem 7 (right)

b / a	3	4	5	6	7	8	14	15	16
1	2,3,3	2,3,4	3,4,5	3,5,6	3,5,7	3,6,8	5,10,17	5,11,18	6,12,19
2	3,4,4	3,5,6	4,6,7	5,7,9	5,8,10	6,9,12	9,17,23	10,18,24	10, 19, 26
3	4,5,6	5,7,8	6,8,9	7,10,12	8,12,14	9,14,16	16,26,32	17,29,35	18,31,37
4		6,9,10	8,11,12	10,14,15	12,16,18	14,19,22	26,41,46	28,45,50	30,49,55
5			11,14,16	13,18,20	16,22,24	19,27,29	40,60,65	43,67,72	47,74,80
6				17,23,25	21,29,31	26,35,38	59,87,93	66,98,104	72,109,116
7					27,37,39	34,46,48	86,123,129	96,139,147	106,156,165
8						43,58,61	119,168,178	136,193,204	152,219,232
14							556,755,820	678,922,1005	817,1113,1219
15								836,1136,1246	1019,1385,1525
16									1254,1704,1886

Off-diagonal Ramsey-like numbers for complete bipartite subgraphs

$R'(a, b, c, d)$ as the minimum number n so that any n-vertex simple undirected graph G must contain a $K_{a,b}$ or its complement G' must contain the complete bipartite graph $K_{c,d}$.

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Theorem

For all $n \in N$ and $0 < p < 1$, if

$$\binom{n}{a}\binom{n-a}{b}p^{ab} + \binom{n}{c}\binom{n-c}{d}(1-p)^{cd} < 1 \tag{7}$$

na, then $R'(a, b, c, d) > n$.

Existence of $R'(a, b)$

Existance proof is achieved by proving following explicit bound.

Theorem

$R'(p, q) \leq R(p + q, p + q)$.

Proof: From Ramsey theorem we know that for any positive integers p and q, $R(p, q)$ always exist.

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As K_{p+q} always contains a subgraph $K_{p,q}$, hence the number that guarantees a monochomatic K_{p+q} always guarantees a monochomatic $K_{p,q}$. □

Upper Bounds on of $R'(a, b)$

Theorem

$R'(a, b) \leq 2^a * R'(a-1, b),\ a < b.$

Theorem

$$if\ \frac{n(n-1)}{2} \geq 2 * \frac{a}{2} * \sqrt[a]{\frac{b-1}{a!}} * n^{2-\frac{1}{a}} + 1, R'(a, b) < n$$

Table 2 : Upper bounds on $R'(a, b)$ from Theorem 13

b / a	1	2	3	4	5	6	7	8
1	2	4	6	8	10	12	14	16
2		11	19	27	35	43	51	59
3			75	111	147	183	219	255
4				516	687	858	1028	1199
5					3339	4172	5005	5839
6						20742	24890	29037
7							125500	146415
8								7456621

Lower bounds for Ramsey like numbers for complete tripartite 3-uniform subgraphs

- An r-uniform hypergraph is a hypergraph where every hyperedge has exactly r vertices. (Hyperedges of a hypergraph are subsets of the vertex set. So, usual graphs are 2-uniform hypergraphs.)

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- $K_{a,b,c}$ is defined as the complete tripartite 3-uniform hypergraph with vertex set $A \cup B \cup C$, where the A, B and C have a, b and c vertices respectively, and $K_{a,b,c}$ has abc 3-uniform hyperedges $\{u, v, w\}$, $u \in A$, $v \in B$ and $w \in C$.

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Lower bounds for Ramsey like numbers for complete tripartite 3-uniform subgraphs

$R'(1,1,1) = 3$; with 3 vertices, there is one possible 3-uniform hyperedge which either is present or absent in G.

$R'(1,1,2) = 4$.
$R'(1,1,3) = 5$.
$R'(1,1,4) = 6$.

Conjecture.

$R'(1,1,b) = b+2$. □

Probabilistic lower bound for $R'(a, b, c)$

Theorem

$$R'(a,b,c) > \frac{(2\pi)^{\left(\frac{3}{2(a+b+c)}\right)} * a^{\left(\frac{a+\frac{1}{2}}{a+b+c}\right)} * b^{\left(\frac{b+\frac{1}{2}}{a+b+c}\right)} * c^{\left(\frac{c+\frac{1}{2}}{a+b+c}\right)}}{e} * 2^{\left(\frac{abc-1}{a+b+c}\right)} \tag{8}$$

Theorem

If $e * 2^{1-abc} * \left(abc\binom{n-3}{a+b+c-3}\binom{a+b+c-3}{b-1}\binom{a+c-2}{c-1} + 1\right) \leq 1,\ R'(a,b,c) > n$

Table 3 : Lower bounds for $R'(a, b, c)$ by Inequality 14(left) and Theorem 15(right)

	a=2	a=3	a=3	a=3	a=4	a=4	a=5	a=6	a=6	a=6	a=6
c / b	5	3	4	5	4	5	5	2	3	4	5
2	9,13	8,11	11,16	16,22	18,25	26,36	40,58	11,16	21,29	36,52	59,87
3	16,22	14,19	23,32	35,50	41,61	68,107	124,208		50,74	107,175	209,371
4	26,36		41,61	68,107	84,138	159,281	334,653			277,521	643,1354
5	40,58			124,208		334,653	800,1765				1740,4194

Significance

- it gives us the minimum number of vertices needed in a graph so that two mutually disjoint subsets of vertices with cardinalities a and b can be guaranteed to have the complete bipartite connectivity property.

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- it gives us the minimum number of vertices needed in a graph so that two mutually disjoint subsets of vertices with cardinalities a and b can be guaranteed to have the complete bipartite connectivity property.
- In the analysis of social networks it may be worthwhile knowing whether all persons in some subset of a persons share b friends.
- In the analysis of transaction systems where either there are many dependent transactions and we need to achieve consistency that either all transactions take place or none of them occur.

Conclusion

The reason behind such Ramsey-type results is that: "The largest partition class always contains the desired substructure".

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- Constructive tighter lower bound for $R'(a, b)$.
- Application of Lovász Local Lemma to Hypergraph Covering Problem.

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