Bounds on Ramsey numbers for complete bipartite and 3-uniform tripartite subgraphs

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of the requirements for the degree of

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by

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Abstract

Ramsey theory is a fundamental, approximately 100-year-old field of mathematics that entails and subsumes various concepts of combinatorics, number theory, geometry, topology, combinatorial geometry, set theory, measure theory, and so on. Ramsey theory is the unification of some ingenious, masterpieces of ideas, and some of its results are among the most beautiful theorems of mathematics. The main mathematical idea of Ramsey theory is this: for any system S, and any large positive integer k, we can choose a large enough supersystem N so that no matter how N is colored with k colors, N always contain a monochromatic copy of S. There are many interpretations of the theory, but we put our focus on the graph interpretation. To be precise, $R(G_1, G_2)$ denote the smallest integer such that for every undirected graph G with $R(G_1, G_2)$ or more vertices, either (i) G contains G_1 as subgraph, or (ii) the complement graph G' of G contains G_2 as subgraph. Traditionally G_1 and G_2 were assumed to be complete graphs and many theories evolved based on these assumptions. So a natural question arises: what happens if G_1 and G_2 are not complete graphs, but rather some other structures? We focus our attention on one such case where G_1 and G_2 are complete bipartite graphs. In particular, $R(K_{a,b}, K_{c,d})$ be the minimum number n so that any n-vertex simple undirected graph G must contain a $K_{a,b}$ or its complement G' must contain a $K_{c,d}$. We demonstrate constructions to show that $R(K_{2,b}, K_{2,b}) > 2b+1$ and $R(K_{2,b}, K_{2,d}) > b+d+1$ for $d \geq b \geq 2$. We establish a lower bound for $R(K_{a,b}, K_{a,b})$ using the probabilistic method that improves over the lower bound given by Chung and Graham [4]. We also establish a lower bound for $R(K_{a,b}, K_{c,d})$ using probabilistic methods. We prove an upper bound for $R(K_{a,b}, K_{a,b})$ and also for the more general case of $R(K_{a,b}, K_{c,d})$ using the Kövári-Sös-Turán theorem. We define $R'(a,b,c)$ to be the minimum number n such that any n-vertex 3-uniform hypergraph $G(V,E)$, or its complement $G'(V,E^c)$ contains a $K_{a,b,c}$. Here, $K_{a,b,c}$ is defined as the complete tripartite 3-uniform hypergraph with vertex set $A \cup B \cup C$, where the A, B and C have a, b and c vertices respectively, and $K_{a,b,c}$ has abc 3-uniform hyperedges $\{u, v, w\}$, $u \in A$, $v \in B$ and $w \in C$. We establish the upper bound of $2b+1$ for $R'(1,1,b)$. We also relate $R'(1,1,b)$ to the existence of a 2-$(2b-1, 3, b-1)$ design. We derive lower bounds for $R'(a,b,c)$ using probabilistic methods.

Keywords: — Ramsey numbers, bipartite graphs, local lemma, probabilistic method, r-uniform hypergraph, t-designs

Chapter 1

Introduction

1.1 Introduction

Ramsey Theory is the branch of mathematics that deals with the basic question of minimum number of elements of a particular structure that must be present so that a particular property holds. Ramsey Theorem states that in any coloring of the edges of a sufficiently large complete graph, one will always find monochromatic complete subgraphs. For example, consider a complete graph of order n; that is, there are n vertices and each vertex is connected to every other vertex by an edge. A complete graph of order 3 is called a triangle. Now color every edge red or blue. How large must n be in order to ensure that there is either a blue triangle or a red triangle?

Paul Erdős who was the leading exponent of Ramsey theory always used to ask two particular questions, first of which explains the concept and the second one emphasizes on the degree of difficulty in solving the problems. The first problem has been named the Party problem. Given 6 people who have been invited to a party can we always find a subset of 3 people all of whom know each other or all of who do not know each other? The problem is equivalent to asking if every coloring of the edges of the complete graph on 6 vertices in the colors black and white contains a subgraph of 3 vertices for which the edges running between these vertices are either all black or all white. The least number of vertices on which the complete graph on these vertices guarantees such a set of 3 vertices is denoted $R(3,3)$. Ramsey type problems typically involve some form of partitioning. In the above example we partitioned the pairs of invitees into 2 sets, those pairs who knew each other and those pairs that did not. Then we asked if we could find 3 pairs in either of the partitions with the property that they formed a triangle of 3 people.

In a typical Ramsey problem we not only insist that our object of interest appear as a substructure of some superstructure, we also ask: how large must our superstructure be so that no matter how we partition it into a given number of parts, one of the partitions contains the desired substructure? Erdős's second problem is that suppose an evil alien would tell mankind to either tell them the value of $R(5,5)$ or they will exterminate the human race, what should they do, what if the aliens asked for value of $R(6,6)$. Erdos says that for $R(5,5)$, It would be best to try to compute it, both by mathematics and with a computer, but if they ask $R(6,6)$, then the best thing would be to destroy them before they destroy us, because we cannot. This is because $R(6,6)$ is known to be lower bounded by 102, so the possible number of graphs is $2^{\binom{102}{2}}$, which is beyond the computational power of any man and machine.

And even more fascinating thing is that for any natural numbers a, b, Ramsey theory proves that $R(a,b)$ always exist. But as it is impossible to compute the exact values of Ramsey numbers for even very small values of a and b, most of the results in this field are non-constructive.

More precisely, Ramsey theorem states that —for any given number of colors c, and any given set of subgraphs $G_1, ..., G_c$, there is a number $R(G_1, ..., G_c)$ such that: if the edges of a complete graph of order $R(G_1, ..., G_c)$(or more) are colored with c different colours, then for some i between 1 and c, it must contain a subgraph G_i whose edges are all color i. This number $R(G_1, ..., G_c)$ is called the Ramsey number for $G_1, ..., G_c$. In particular, $n = R(K_a, K_b)$ is the smallest number of vertices so that any undirected graph G with n or more vertices contains either a K_a or an independent set of size b. We know that $R(K_3, K_3) = 6$, $R(K_3, K_4) = 9$, $R(K_4, K_4) = 18$, $R(K_4, K_5) = 25$, $R(K_3, K_8) = 28$ and $R(K_3, K_9) = 36$ (see [12, 13]).

Ramsey theorem is a foundational result in combinatorics. Ramsey theory seeks regularity amid disorder: general conditions for the existence of substructures with regular properties. In the application above, it is a question of the existence of monochromatic subsets, that is, subsets of connected edges of just one colour. Hence, there are many variants of Ramsey type structure. We focus on a specific kind Ramsey theory where we try to find the complete bipartite subgraphs and multipartite hypergraphs within colorings of complete graphs and compelete hypergraphs.

1.2 Our Contribution and Significance

We define $R(K_{a,b}, K_{a,b})$ as the minimum number n of vertices so that any n-vertex simple undirected graph G or its complement G' must contain the complete bipartite graph $K_{a,b}$. Equivalently, $R(K_{a,b}, K_{a,b})$ is the minimum number n of vertices such that any bicoloring of the edges of the n-vertex complete undirected graph K_n would contain a monochromatic $K_{a,b}$. Again for the more general case, let $R(K_{a,b}, K_{c,d})$ be the minimum number n so that any n-vertex simple undirected graph G must contain a $K_{a,b}$ or its complement G' must contain a $K_{c,d}$. We refer to $R(K_{a,b}, K_{a,b})$ as unbalanced diagonal case and $R(K_{a,b}, K_{c,d})$ as unbalanced off-diagonal case. As stated earlier, finding the exact values of these numbers is really difficult, hence we use some constructive techniques for smaller values and probabilistic methods for the general case and establish some lower bounds. We define $R'(a,b,c)$ to be the minimum number n such that any n-vertex 3-uniform hypergraph $G(V,E)$, or its complement $G'(V,E^c)$ contains a $K_{a,b,c}$. We establish a upper bound on $R'(1,1,b)$, relate $R'(1,1,b)$ to the existence of a 2-$(2b-1,3,b-1)$ design and derive lower bounds for $R'(a,b,c)$ using probabilistic methods.

The significance of such a number is that it gives us the minimum number of vertices needed in a graph so that two mutually disjoint subsets of vertices with cardinalities a and b can be guaranteed to have the complete bipartite connectivity property as mentioned. In the analysis of social networks it may be worthwhile knowing whether all persons in some subset of a persons share b friends, or none of the a persons of some other subset share friendship with some set of b persons. This can also be helpful in the analysis of dependencies, where there are many entities in one partite, which are all dependent on entities in the other partite; we need to achieve consistencies that either all dependencies exist between a pair of two partites, or none of the dependencies exist between possibly another pair of two partites. These Ramsey numbers are different from the usual Ramsey numbers $R(K_a, K_b)$, where instead of complete bipartite subgraphs, we look for the existence of complete subgraphs. The bipartite version of the party problem may be stated like this: Given 6 people who have been invited to a party can we always find a subset of 2 people all of whom know some other group of 2 people or

2 people both of whom are strangers with some other group of 2 people ? In fact we always can. This is because the above problem actually asks us whether $R(K_{2,2}, K_{2,2}) \leq 6$ and it is known that $R(K_{2,2}, K_{2,2})$ is equals to 6.

1.3 Organization of the Thesis

The cases we study for bipartite graphs as subgraphs are (i) the *unbalanced diagonal case* for $R(K_{a,b}, K_{a,b})$, (ii) the *balanced off-diagonal case* for $R(K_{a,a}, K_{b,b})$, and (iii) the *unbalanced off-diagonal case* for $R(K_{a,b}, K_{c,d})$. The thesis is broken into several Chapters. In Chapter 2, we do a survey of some of the significant earlier works on this field. In Chapter 3, we consider the *unbalanced diagonal case* i.e. Ramsey numbers $R(K_{a,b}, K_{a,b})$, derive the value of $R(K_{1,b}, K_{1,b})$, lower bounds for $R(K_{2,b}, K_{2,b})$ for small values of b, a combinatorial constructive lower bound for $R(K_{2,b}, K_{2,b})$ and lower bounds for the arbitrary a, b using probabilistic methods. In Chapter 4, we consider the *balanced off-diagonal case* i.e. Ramsey numbers $R(K_{a,b}, K_{c,d})$, derive constructive lower bound for $R(K_{2,b}, K_{2,d})$, lower bounds for the arbitrary a, b, c, d using probabilistic methods and extend the results for the *balanced off-diagonal case* i.e. for Ramsey numbers $R(K_{a,a}, K_{b,b})$. In Chapter 5, we prove the existence of such numbers for all natural numbers a, b and perform upper bound analysis of $R(K_{a,b}, K_{a,b})$ and $R(K_{a,b}, K_{c,d})$. In Chapter 6 we extend similar methods for 3-uniform tripartite hypergraphs, deriving lower bounds for the Ramsey numbers $R'(a, b, c)$. Here, $R'(a, b, c)$ is the minimum number n such that any n-vertex 3-uniform hypergraph $G(V, E)$, or its complement $G'(V, E^c)$ contains a $K_{a,b,c}$. Here, $K_{a,b,c}$ is defined as the complete tripartite 3-uniform hypergraph with vertex set $A \cup B \cup C$, where the A, B and C have a, b and c vertices respectively, and $K_{a,b,c}$ has abc 3-uniform hyperedges $\{u, v, w\}$, $u \in A$, $v \in B$ and $w \in C$. In Chapter 7 we conclude with a few remarks and future research directions.

Chapter 2

Preliminaries and Existing Results

A *graph* is an abstract representation of a set of objects where some pairs of the objects are connected by links. The interconnected objects are represented by vertices, and the links that connect some pairs of vertices are called edges. Formally a graph G is represented as $G(V,E)$, where V is the set of vertices of the graph, E is the set of edges of the graph. Each edge in E is represented as (u,v), where $u,v \in V$ and u,v are the vertices between which the edge is present. A graph is undirected if all the edges in the graph are undirected. A graph is simple if it does not contain any self loops or parallel edges. Through out the thesis, graph stands for a simple, undirected graph. Hypergraph is a generalization of a graph in which an edge can connect any number of vertices. Formally a hypergraph H is a pair $H = (V,S)$, where V is the set of vertices of the graph, S is the set of hyperedges of the graph, each hyperedge being some non-empty collection of vertices. Hence $S \subseteq POWERSET(V) - \phi$. A Hypergraph is called r-uniform if all the hyperedges present in the graph consists of r vertices.

Edge coloring is the assignment of colors to all the edges in the graph. For example, in edge coloring with two colors, we color all the edges using two colors. Throughout the thesis, coloring stands for edge coloring.

2.1 Pigeonhole principle

Definition 1. *The Pigeonhole principle states that if m items are put into n pigeonholes* *with $m > n$, then at least one pigeonhole must contain more than one item. In general, if $mn+1$ items are placed in n pigeonholes, then there will be at least one pigeonhole with $m+1$ or more items in it.*

The Pigeonhole Principle sounds trivial but its applications are not so subtle in most of the cases. Many of the arguments given for proving bounds on Ramsey numbers use this principle(eg. the proof of theorem $R(3,3) = 6$ uses this, see section below). One such ramsey type result given by Erdős and Szekeres that is a direct consequence of pigeonhole principle detailed below.

Theorem 1 (Erdős and Szekeres). *Given a sequence of $mn+1$ distinct real numbers, if it does not contain a monotone increasing subsequence of length $m+1$ then it must contain a monotone decreasing subsequence of length $n+1$.*

Proof We prove the statement by contradiction. Let $S := \{a_1, ..., a_{mn+1}\}$ be a sequence of $mn+1$ distinct real numbers. Assume that the result is false i.e. assume S contains neither a monotone increasing subsequence of length $m+1$ nor a monotone decreasing subsequence of length $n+1$.

For each number a_k in the sequence, form the ordered pair (i_k, j_k), where i_k is the length of the longest increasing subsequence beginning with a_k, and j_k is the length of the longest decreasing subsequence ending with a_k. Then, since the result is false, $1 \leq i_k \leq m$ and $1 \leq j_k \leq n$. Thus we have $mn+1$ ordered pairs, of which at most mn are distinct. Hence by pigeonhole principle, two members of the sequence, say x and y, are associated with the same ordered pair (s,t). Without loss of generality we may assume that x precedes y in the sequence.

If $x < y$, then x, together with the longest increasing subsequence beginning with y, is an increasing subsequence of length $(s+1)$, contradicting the fact that s is the length of the longest increasing subsequence beginning with x. Hence $x \geq y$. But then, y, together with the longest decreasing subsequence ending with x, is a subsequence of length $(t+1)$, contradicting that the longest decreasing subsequence ending with y is of length t. Hence our assumption has to be false, and the result is therefore true. $\square$

2.2 Ramsey theory: Pre-history and early history

There are some significant results preceding the birth of Ramsey theory and of the early days. Each one of these results is special in its own right and suggest concepts fundamental to extremal theory. We discuss only some of the selected results that in the order they originated.

2.2.1 David Hilbert's cube lemma 1892

This was the first Ramseyan result published by David Hilbert in 1892, merely as a tool for his study of irreducibility of rational functions with integral coefficients. A set $Q_n(a, x_1, x_2, ..., x_n)$ of integers is called an *n-dimensional affine cube* if there exists $n+1$ positive integers $a, x_1, x_2, ..., x_n$ such that

$$Q_n(a, x_1, x_2, ..., x_n) = \{a + \sum_{i \in F} x_i : \phi \neq F \subset \{1, 2, ..., n\}\} \tag{2.1}$$

Let us denote the starting segment of positive integers {1,2,...,n} as $[n]$. Now the lemma can be stated as follows.

Lemma 1 (The Hilbert's Cube Lemma). *For every pair of positive integers r, n, there exists a least positive integer $m = H(r,n)$ such that in every r coloring of a $[m]$, there exists a monochromatic r-affine cube.*

It seems that David Hilbert's monochromatic cube lemma was the first example of Ramseyan mathematics. Apparently nobody, including Hilbert, appreciated the lemma much. Hilbert did not continue research in the direction the lemma showed. The lemma was added as the first instance of Ramseyan thought, but failed to influence the field much at that time.

2.2.2 The Issai Schur Theorem 1916

Issai Schur, in his pioneering paper Über Die Congruenz $x^m + y^m \equiv z^m \mod p$, created, as he puts it "a very simple lemma, that belongs to combinatorics more than number theory". The proof stated uses another lemma stated below.

Lemma 2 (R. E. Greenwood and A. M. Gleason 1955). *For an positive integer n, there exists a positive integer $S(n)$ such that any n coloring of edges of $K_{S(n)}$ would always contain a monochromatic triangle K_3.*

Theorem 2 (Issai Schur). *For any positive integer n there exists a positive integer $S(n)$ such that any n coloring of the set $[S(n)]$ contains integer a, b, c of same color such that $a + b = c$.*

Proof Let all the positive integers be colored with the n colors. By Theorem 2, there exists a $S(n)$ such that edge coloring of $K_{S(n)}$ would always contain a monochromatic K_3. Construct the complete graph $K_{S(n)}$ with $[S(n)] = \{1,2,...,S(n)\}$ as vertices. Color the edge (i, j) with the color of $|i-j|$th number, which were colored earlier. So we get a complete graph $K_{S(n)}$ whose edges are coloured with n colors. By Theorem 2, there exists a monochromatic K_3. Let it be $\{i, j,k\}$ with $i > j > k$, whose edges ij, jk, ik are coloured with same color.

Let $a = i-j$, $b = j-k$ and $c = i-k$. As the edges are colored in the same color, a,b,c must be colored in same color in the original coloring. Now $a+b = i-j+j-k$ $= i-k = c$. Hence theorem is proved. □

2.2.3 The Baudet–Schur–Van der Waerden Theorem 1927

In trying to prove his own conjectures, Schur realized he needs to conjecture another simple lemma. But Baudet created the same conjecture independently, which was proven by Bartel Leendert van der Waerden in 1927. This opened the path way for many more results and marked the beginning of another theory.

Theorem 3 (Baudet–Schur–Van der Waerden). *For any k,l, there is a $W = W(k,l)$ such that any k coloring of the set $[W]$ always contain a $l-$term monochormatic arithmetic progression.*

2.2.4 Original Ramsey principles

in 1928, Frank Ramsey submitted a paper that got published after his death in 1930. This paper contained a infinite and a finite version of the theorem that is since than being referred to as *Ramsey* Theorem.

Theorem 4 (Infinite Ramsey 1930). *For any positive integers k and r, if the collection of all r-element subsets of an infinite set S is colored in k colors, then S contains an infinite subset S_1 such that all r-element subsets of S_1 are assigned the same color.*

Theorem 5 (Finite Ramsey 1930). *For any positive integers r, n, and k there is an integer $m_0 = R(r,n,k)$ such that if $m \geq m_0$ and the collection of all r-element subsets of an m-element set S_m is colored in k colors, then S_m contains an n-element subset S_n such that all r-element subsets of S_n are assigned the same color.*

This was a evolutionary result as all the previous results were confined to a particular setting, but this was the most generic one and opened the ways for various new results.

In the thesis, we use the graph interpretation of the finite Ramsey theorem. Following theorem uses Ramsey numbers as a tool for proving the existence of another interesting result.

2.2.5 Happy end problem

Theorem 6 (Erdős and Szekeres). *For any integer $n \geq 3$, there exists a smallest positive integer $N(n)$ such that any set of at least $N(n)$ points in general position in the plane (i.e., no three of the points are on a line) contains n points that are the vertices of a convex n-gon.*

The following simple proof is due to Micheal Tarsi which uses the Ramsey numbers to bound $N(n)$.

Proof Let $n \geq 3$ be a positive integer. By the Ramsey principle 5 ($r = 3$ and $k = 2$) there is an integer $m_0 = R(3,n,2)$ such that, if $m > m_0$ and the collection of all 3-element subsets of an m-element subset S_m are colored in two colors, then S_m contains an n-element subset S_n such that all 3-element subsets of S_n are assigned the same color. Let now S_m be a set of m points in the plane in general position labelled with integers $1,2,...,m$. We color a 3-element set $\{i, j,k\}$ where $i < j < k$, red if we travel from i to j to k in a clockwise direction, and blue if counter-clockwise. By the assertion above, S_m contains a n-element subset S_n such that all 3-element subsets of S_n are assigned the same color, that is, have the same orientation. But this means precisely that S_n forms a convex n-gon. Hence $N(n) \leq R(3,n,2)$. □

Erdős and Szekeres proved that for any positive integer $n > 3$, $2^{n-2} < N(n) \leq \binom{2n-4}{n-2} + 1$ and they conjectured that for any positive integer $n > 3$, $N(n) = 2^{n-2} + 1$. This conjecture is called as Happy end conjecture and is open till date.

From now onwards, we focus only on the graph interpretation of finite Ramsey theorem. We first define the numbers and discuss some of the results from earlier works.

2.3 Some basic results for Ramsey numbers

$R(K_a, K_b)$ is the smallest number of vertices so that any undirected graph G with $R(K_a, K_b)$ or more vertices contains either a K_a or its complement graph G' contains a K_b. It follows from the definition of Ramsey Theorem that for positive integers a and b,

$R(K_a, K_b) = R(K_b, K_a)$. It is not that difficult to note $R(K_1, K_k) = 1$, as any graph with 1 vertex trivially contains a K_1.

Theorem 7. *$R(K_2, K_b) = b$ for all $b \geq 2$.*

Proof Choose some positive integer $b \geq 2$. First we will show that $R(K_2, K_b) > b - 1$ by constructing a 2-coloring on K_{b-1} that contains neither a red K_2 nor a blue K_b. The coloring in which every edge is blue satisfies these requirements. It certainly does not contain a red K_2 and cannot possibly contain a blue K_b as we have $b - 1$ vertices, so $R(K_2, K_b) > b - 1$.

Next, suppose that the edges of K_b are 2-colored in some fashion. If any of the edges are red, then K_b will contain a red K_2. If none of the edges are red, then we are left with a blue K_b. So $R(K_2, K_b) \leq b$. Thus, we can conclude that $R(K_2, K_b) = b$ for all $b \geq 2$. □

Theorem 8. *$R(K_3, K_3) = 6$.*

Proof Consider any 2-coloring on K_6. Let the vertices be v, x, y, z, a, b. Choose some vertex v from the graph. Because there are 5 edges incident to v, by the pigeon hole principle, at least three of these edges must be the same color. We will call them (v, x), (v, y) and (v, z), and we assume they are all red. If at least one of the edges out of (x, y), (x, z), or (y, z) is red, then we have a red K_3. If none of these is red, then we have a blue K_3. For example, If (x, y) is red, (v, x), (v, y) and (x, y) form a red K_3. If none of the edges out of (x, y), (x, z) or (y, z) are red, that means they are all blue and they form a blue K_3. Thus, $R(K_3, K_3) \leq 6$.
Next, consider the 2-coloring on K_5 coloring a C_5 by red and the other C_5 (that is the complement of first C_5)by blue . This coloring does not contain a monochromatic K_3 in either red or blue, so we know that $R(K_3, K_3) > 5$. Thus, $R(K_3, K_3) = 6$. □

Theorem 9. *The Ramsey numbers are monotone.*

Proof Let $a_1 \geq a_2$ and $b_1 \geq b_2$ then if n is large enough to guarantee the existence of either a red K_{a_1} or an blue K_{b_1} then n also guarantees the existence of a red K_{a_2} or an blue K_{b_2} , as K_{a_2} is a subgraph of K_{a_1} and K_{b_2} is a subgraph of K_{b_1} . But If we consider the other side, then existence of K_{a_2} does not guarantee existence of K_{a_1} and existence of K_{b_2} does not guarantee existence of K_{b_1}. Hence $R(K_{a_1}, K_{b_1}) \geq R(K_{a_2}, K_{b_2})$. □

Theorem 10. *For all natural numbers a, b, $R(K_a, K_b)$ always exists and is always finite. If $a \geq 2$ and $b \geq 2$, then $R(K_a, K_b) \leq R(K_{a-1}, K_b) + R(K_a, K_{b-1})$.*

Proof We will proceed by induction on $a+b$.

First we consider the base case in which $a+b$ = 2. The only way this can be true is if $a = b = 1$, and it is clear that $R(1,1)$ = 1. Now we assume that the theorem holds whenever $a+b < N$, for some positive integer N. Let P and Q be integers such that $P+Q = N$. Then $P+Q-1 < N$, so by our assumption we know that $R(P-1,Q)$ and $R(P,Q-1)$ exists i.e. both terms are finite.

Consider a complete graph on $R(P-1,Q)$ + $R(P,Q-1)$ vertices.

- Pick a vertex v from the graph, and partition the remaining vertices into two sets M and N, such that for every vertex w, w is in M if (v,w) pair is blue, and w is in N if (v,w) is red.

- Because the graph has $R(P-1,Q)+R(P,Q-1) = |M|+|N|+1$ vertices,By symmetry, it follows that either

$$|M| \geq R(P-1,Q) \text{ or } |N| \geq R(P,Q-1)$$

In the former case, if M has a red K_Q then so does the original graph and we are finished. Otherwise M has a blue K_{P-1} and so $M \cup \{v\}$ has blue K_P by definition of M. In the latter case, if N has a blue K_P then so does the original graph and we are finished. Otherwise N has a red K_{Q-1} and so $N \cup \{v\}$ has red K_q by definition of N. □

Theorem 11. *For natural numbers a and b,*

$$R(K_a, K_a) > 2^{\binom{a-1}{2}} \tag{2.2}$$

Proof The proof uses probabilistic analysis that was originally devised by Paul Erdós. We want some n (ideally as large as possible) so that we can somehow colour the edges of K_n using two colors (say red and blue) in such a way that we get neither a red K_a or a blue K_a. He then developed a non-constructive method of choosing a very large n satisfying the constraints.

Let n be the number of vertices of graph G. Then the total number of distinct K_a possible is $\binom{n}{a}$. Each K_a has exactly $\binom{a}{2}$ edges. Each edge can be either of color 1 or color 2 with equal probability. So probability of a particular K_a of color 1 is $\left(\frac{1}{2}\right)^{\binom{a}{2}}$. So probability that a particular K_a of either color 1 or color 2 exists is $2\cdot 2^{-\binom{a}{2}} = 2^{1-\binom{a}{2}}$. So probalility p of any monochromatic $K_a = \binom{n}{a}\cdot 2^{1-\binom{a}{2}}$.

Our objective is to choose as large n as possible with $p < 1$. So choosing $n = 2^{\left(\frac{a-1}{2}\right)}$, we get

$$\begin{aligned}
p &= \binom{n}{a}\cdot 2^{1-\binom{a}{2}} \\
&\leq \frac{n^a}{a!}\cdot 2^{1-\binom{a}{2}} \\
&< 2^{a\left(\frac{a-1}{2}\right)}\cdot 2^{-\binom{a}{2}} \\
&= 1.
\end{aligned}$$

As the probability p is strictly less than 1, it guarantees our requirement and thus gives a lower bound. □

Extension to Hypergraphs:

For any integers m and c, and any integers $n_1,...,n_c$, there is an integer $R(n_1,...,n_c;c,m)$ such that if the hyperedges of a complete m-hypergraph of order $R(n_1,...,n_c;c,m)$ are coloured with c different colours, then for some i between 1 and c, the hypergraph must contain a complete sub-m-hypergraph of order n_i whose hyperedges are all colour i. There are only few results available for multicolor hypergraph Ramsey numbers, most significant one is that $R(3,3,3) = 17$.

Ramsey theorem though just focuses on complete substructure, in hind sight, it also generates prospect of existence of other substructures within large superstructures. This gives motivation to look for bipartite and multipartite subgraphs, which is the center of attention of the thesis.

2.4 Definition of $R(K_{a,b},K_{a,b})$, $R(K_{a,b},K_{c,d})$ and $R'(a,b,c)$ and previous work

$R(K_{a,b},K_{a,b})$ as the minimum number n of vertices so that any n-vertex simple undirected graph G or its complement G' must contain the complete bipartite graph $K_{a,b}$.

Chvátal and Harary [3] were the first to show that $R(C_4,C_4) = 6$. As $K_{2,2}$ is identical to C_4, $R(K_{2,2},K_{2,2}) = 6$. Some similar results are $R(K_{2,3},K_{2,3}) = 10$ [1], $R(K_{2,4},K_{2,4}) = 14$ [7], and $R(K_{2,5},K_{2,5}) = 18$ [7]. Lortz and Mengersen [9] conjectured that $R(K_{2,b},K_{2,b}) \geq 4b-3$, for all $b \geq 2$. Exoo et al. [7] proved that $R(K_{2,b},K_{2,b}) \leq 4b-2$ for all $b \geq 2$, where the equality holds if and only if a strongly regular $(4b-3, 2b-2, b-2, b-1)$-graph exists. There are many such results for $R(K_{a,b},K_{a,b})$ for various values of a and b in [12].

$R(K_{a,b},K_{c,d})$ be the minimum number n so that any n-vertex simple undirected graph G must contain a $K_{a,b}$ or its complement G' must contain a $K_{c,d}$. Harary proved that $R(K_{1,n},K_{1,m}) = n+m-x$, where $x = 1$ if both n and m are even and $x = 0$ otherwise [8]. H. Harborth and I. Mengersen proved that $R(K_{1,3},K_{m,n}) = m+n+2$ for $m,n \geq 1$ [16]. Chen et. al. showed that $R(K_{1,n+1}K_{2,2}) \leq R(K_{1,n+}K_{2,2})+2$[17]. $R(K_{2,n-1},K_{2,n}) \leq 4n-4$ for all $n \geq 3$, with the equality if there exists a symmetric Hadamard matrix of order $4n-4$. There are only 4 cases in which the equality does not hold for $3 \leq n \leq 58$, namely 30, 40, 44 and 48[18].

We define $R'(a,b,c)$ be the minimum number n such that any n-vertex 3-uniform hypergraph $G(V,E)$, or its complement $G'(V,E^c)$ contains a $K_{a,b,c}$. Here, $K_{a,b,c}$ is defined as the complete tripartite 3-uniform hypergraph with vertex set $A \cup B \cup C$, where the A, B and C have a, b and c vertices respectively, and $K_{a,b,c}$ has abc 3-uniform hyperedges $\{u,v,w\}$, $u \in A$, $v \in B$ and $w \in C$. There are no known results to this problem to the best our knowledge.

Chapter 3

The unbalanced diagonal case : $R(K_{a,b}, K_{a,b})$

It follows from the definition that for positive integers a and b, $R(K_{a,b}, K_{a,b})$ = $R(K_{b,a}, K_{b,a})$. From the definition of $R(K_{a,b}, K_{a,b})$, it is clear that $R(K_{1,1}, K_{1,1}) = 2$ and $R(K_{1,2}, K_{1,2}) = 3$. To show that $R(K_{1,3}, K_{1,3}) \geq 6$, observe that we need at least 4 vertices and neither a 4-cycle nor it complement has a $K_{1,3}$. Further, observe that neither a 5-cycle in K_5, nor its complement (also a 5-cycle) has a $K_{1,3}$. We now prove that $R(K_{a,b}, K_{a,b})$ increases monotonically.

Theorem 12. *$R(K_{a,b}, K_{a,b})$ are monotone, i.e for $a_1 \geq a_2$ and $b_1 \geq b_2$, $R(K_{a_1,b_1}, K_{a_1,b_1}) \geq R(K_{a_2,b_2}, K_{a_2,b_2})$.*

Proof Let $a_1 \geq a_2$ and $b_1 \geq b_2$ then if n is large enough to guarantee the existence of either a red K_{a_1,b_1} or an blue K_{a_1,b_1} then n also guarantees the existence of a red K_{a_2,b_2} or an blue K_{a_2,b_2} , as K_{a_2,b_2} is a subgraph of K_{a_1,b_1} . But If we consider the other side, then existence of K_{a_2,b_2} does not guarantee existence of K_{a_1,b_1} as the latter is a larger structure than the previous. Hence $R(a_1, b_1) \geq R(a_2, b_2)$. □

Now we prove the exact values of $R(K_{1,b}, K_{1,b})$ (This result was originally given by Burr and Roberts [2], but the analysis below is entirely independent).

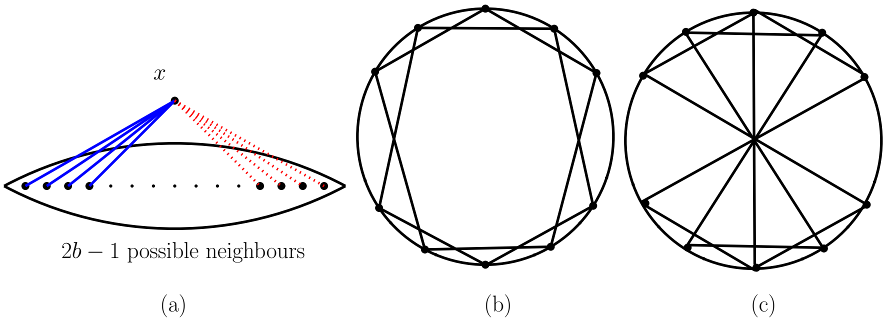

Figure 3.1: (a) pigeonhole explanation (b) $b-1=2m$ for $b=5$ (c) $b-1=2m+1$ for $b=6$

3.1 $R(K_{1,b},K_{1,b}) = 2b$, *if b is odd and 2b-1 if b is even*

Theorem 13. *$2b-1 \leq R(K_{1,b},K_{1,b}) \leq 2b$.*

Proof $R(K_{1,b},K_{1,b}) \leq 2b : n = 2b$ vertices:
for any vertex x, there are exactly $2b-1$ possible neighbours, so by pigeon hole principle, x must contain b neighbours in atleast one of G or G'. Those b neighbours combined with x forms the $K_{1,b}$.

$R(K_{1,b},K_{1,b}) \geq 2b-1 : n = 2b-2$(i.e $< 2b-1$) vertices:
To show that $R(K_{1,b},K_{1,b}) \geq 2b-1$, we need to give a general construction with $2b-2$ vertices graphs G and G' free from $K_{1,b}$. So our construction would generate a graph G that is $(b-1)$-regular(that will be obviously free from $K_{1,b}$), such that the number of possible neighbours for any vertex in G' cannot exceed $b-1$.

Construction of G: If $b-1=2m$ is even, put all the vertices around a circle, and join each to its m nearest neighbors on either side.If $b-1=2m+1$ is odd (and as $n=2b-2$ is even), put the vertices on a circle, join each to its m nearest neighbors on each side, and also to the vertex directly opposite. The entire construction is illustrated for specific values in Figure 3.1. This will result in a $(b-1)$-regular graph G such that G and its complement G' are free from $K_{1,b}$. □

Theorem 14. *$R(K_{1,b},K_{1,b}) = 2b$, if b is odd.*

Proof To proof this, all we need to show is with $n=2b-1$ vertices, there exists a graph G such that both G and its complement G' are free from $K_{1,b}$. As b is odd, $b-1$

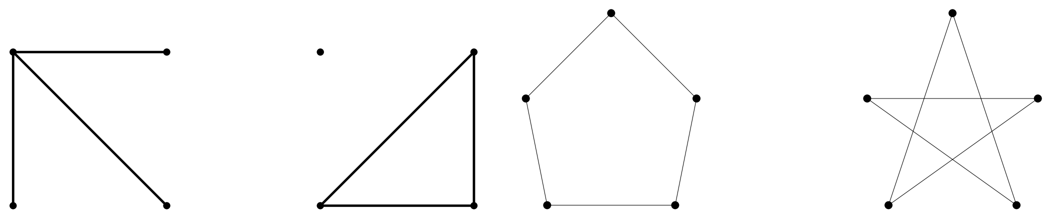

Figure 3.2: G and G' with n = 4 and n = 5 without a $K_{2,2}$

is even. If $b-1 = 2m$, put all the vertices around a circle, and join each to its m nearest neighbors on either side to construct G that is a $(b-1)$-regular graph, hence free from $K_{1,b}$. As a result, the number of possible neighbours for any vertex in G' is exactly $b-1$, hence G' is also free from $K_{1,b}$. □

Theorem 15. $R(K_{1,b}, K_{1,b}) = 2b-1$*, if b is even.*

Proof To proof this, we will show that any graph G or its complement G' with $2b-1$ vertices will always contain $K_{1,b}$. Any vertex with degree b means that there exist some $K_{1,b}$. Hence to avoid $K_{1,b}$, the degree of any vertex cannot exceed $b-1$. Again if some vertex has degree $b-2$ in G, then that means the vertex has $2b-2-(b-2) = b$ neighbours in G', which introduces the $K_{1,b}$. Hence to avoid $K_{1,b}$ in both G and G', the only possible solution is that both G and G' are $(b-1)$-regular.

As b is even, $b-1$ is odd and hence it is impossible to construct a $(b-1)$-regular graph with $2b-1$ verices(this is clear from the degree sum formula which proves that the sum of degrees of vertices of any graph is always even and is equal to $2e$, where e is total number of edges in the graph and as both $b-1$ and $2b-1$ are odd, the degree sum $(b-1)(2b-1)$ is also odd).

This means that there always exist atleast one vertex with degree less than or greater than $b-1$ in G. G' has the $K_{1,b}$ in the first case where as G has the $K_{1,b}$ in the latter. □

3.2 Lower bounds for $R(K_{2,b}, K_{2,b})$ for small values of b

Theorem 16. $R(K_{2,2}, K_{2,2}) > 5$

Proof By Counter Example: To establish the bound, we show graphs with n = 4 and n = 5 free from $K_{2,2}$ in 3.2. □

Theorem 17. $R(K_{2,2}, K_{2,2}) = 6$.

Proof Let us have 6 vertices namely x_1 through x_6. So any vertex say x_1 has 5 possible neighbours.So by pigeon hole principle, x_1 has atleast 3 neighbours in either G or G'. Without loss of generality we assume that the graph in which x_1 has 3 neighbours is G. Let its neighbours be x_2,x_3,x_4. For x_5 and x_6, there are two possibilities.

Case 1: *atleast one of x_5, x_6 has > 3 neighbours in G.*

Let that vertex be x_5. As x_5 has > 3 neighbours, it has atleast two neighbours in the neighbour set $\{x_2, x_3, x_4\}$ of x_1. And hence $\{ x_1, x_2\}$ form one partite set and their common neighbours form the other partite set to give a $K_{2,2}$.

Case 2: *both x_5, x_6 has ≤ 2 neighbours in G.*

Case 2.1: *x_5, x_6 are non-neighbours and atleast one of them has two neighbours.*

In this case, the vertex with two neighbours combined with x_1 forms one partite set and their common neighbours form the other partite set.

Case 2.2: x_5, x_6 are neighbours and atleast one of them has two neighbours.

In this case, there is atleast 1 vertex from the set $\{x_2, x_3, x_4\}$ which is not a neighbour of both x_5 and x_6. Let that vertex be x_3. In the complement graph, x_5 and x_6 form one partite set and x_1 and x_3 form the other partite set to give a $K_{2,2}$.

Case 2.3: *both x_5 and x_6 have < 2 neighbours.*

As there are 5 possible neighbours for both x_5 and x_6, and as both x_5 and x_6 has less than or equal to 1 neighbours in G, they must have atleast 3 common neighbours in G', thus forming a $K_{2,2}$ with x_5, x_6 as one partite set and their common neighbours form the other partite set. □

As $K_{2,2}$ is also a C_4, this result implies that with number of vertices greater than equal to 6, we are guaranteed to find a C_4(i.e. $K_{2,2}$) in either G or complement of G.

Theorem 18. $R(K_{2,3}, K_{2,3}) > 7$.

Proof By Counter Example:

To establish the bound, we show graphs with $n = 7$ free from $K_{2,3}$ in 3.3 . □

Theorem 19. $R(K_{2,4}, K_{2,4}) > 9$.

Proof By Counter Example:

To establish the bound, we show graphs with $n = 9$ free from $K_{2,4}$ in 3.4 .

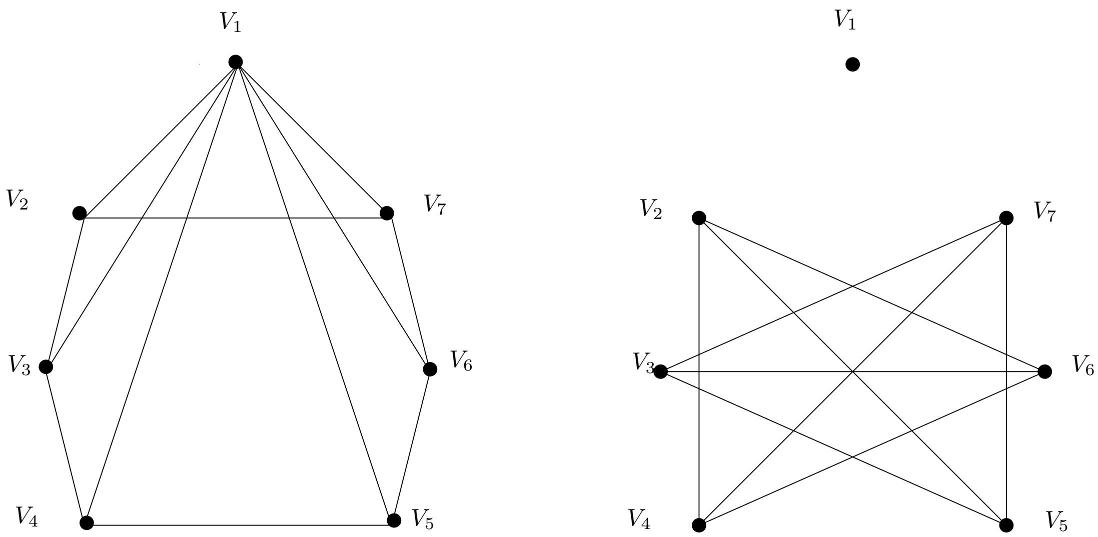

Figure 3.3: G and G' with $n = 7$ without a $K_{2,3}$

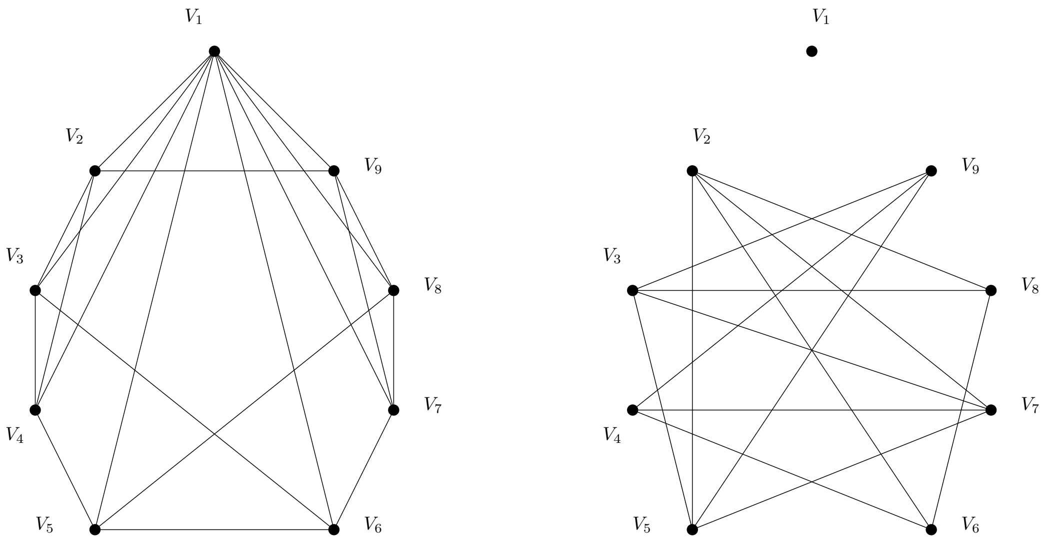

Figure 3.4: G and G' with $n = 9$ without a $K_{2,4}$

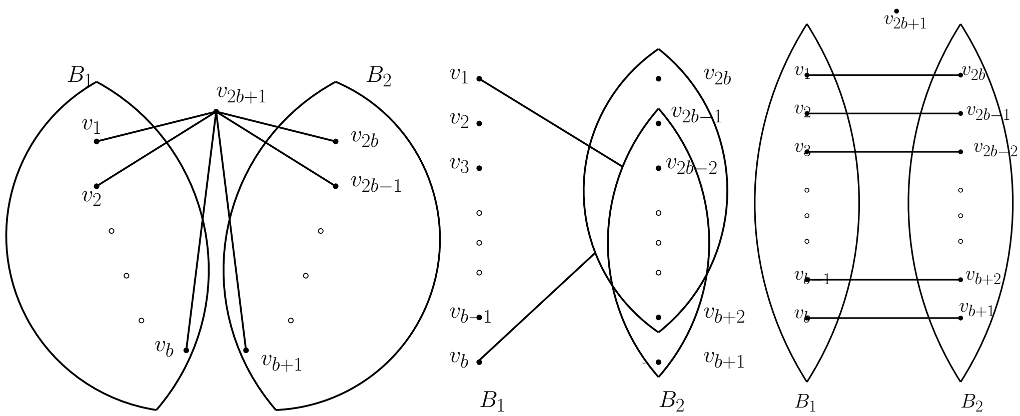

Figure 3.5: Construction of G(left two): generation of B_1, B_2 and addition of edges. Resulting G'(rightmost): In G', B_1 and B_2 become K_b, and B_1 and B_2 have a perfect matching.

These counter examples can be generated by brute force technique and the code for generating a counterexample for $R(K_{2,3}, K_{2,3}) > 7$ is attached in the Appendix B.1. $\square$

The exact values for the above numbers are known and are given in [12].

3.3 A constructive lower bound for $R(K_{2,b}, K_{2,b})$

The following lower bound for $R(K_{2,b}, K_{2,b})$ involves an explicit construction as follows.

Theorem 20. *$R(K_{2,b}, K_{2,b}) > 2b+1$, for all integers $b \geq 2$.*

Proof For $b \geq 2$, we show that with $2b+1$ vertices, there always exist a graph G such that both G and its complement G' do not contain $K_{2,b}$. The entire construction is illustrated in Figure 3.3. Let the vertices be labelled v_1, v_2, ..., v_{2b+1}. Connect v_{2b+1} to every other vertex. In order to avoid $K_{2,b}$, no other vertex out of v_1, v_2, ..., v_{2b} should be connected to b or more vertices in the set v_1, v_2, ..., v_{2b}. So, any of these $2b$ vertices can have a maximum of $b-1$ neighbours other than v_{2b+1}. We distribute these $2b$ vertices into two groups, keeping v_1, v_2, ..., v_b in one group B_1, and v_b+1, v_b+2, ..., v_{2b} in the other group B_2. Now every vertex from B_1 can be connected to at most $b-1$ vertices from B_2 such that we can still avoid $K_{2,b}$. There are $\binom{b}{b-1} = b$ such distinct groups of size $b-1$ in B_2. Now each vertex of B_1 is connected to one such distinct group of size

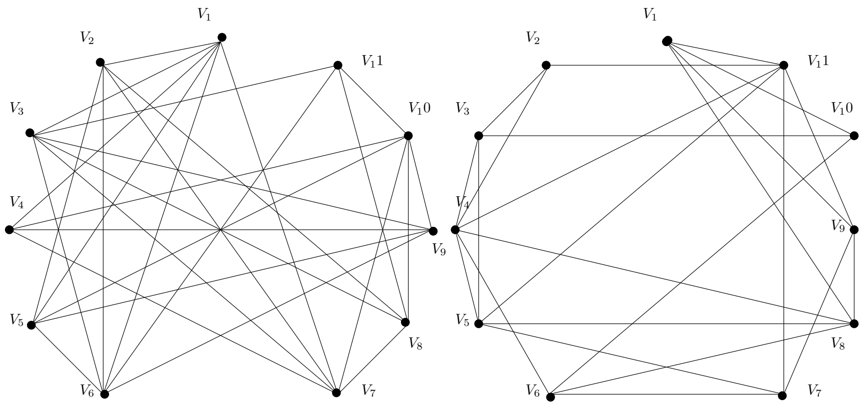

Figure 3.6: G and G' with n = 11 without a $K_{3,3}$

$b-1$ from B_2. We claim the degree of every vertex except v_{2b+1} is b. Firstly, every vertex of B_1 is connected to $b-1$ vertices of B_2, and the single vertex v_{2b+1}. Secondly, every vertex of B_2 (i) is connected to v_{2b+1}, and (ii) also present in exactly $b-1$ separate groups, where each group is connected to exactly one vertex of B_1. So, every vertex of $B_1 \cup B_2$ has degree b. Therefore G is $K_{2,b}$-free.

Now Consider G'. Since v_{2b+1} is connected to every other vertex in G, it is isolated in G'. So, the number of possible neighbours for vertices v_1, v_2, ..., v_{2b+1} becomes $2b-1$. Since each vertex in G is connected to $b-1$ vertices other than v_{2b+1}, the number of possible neighbours for each vertex is restricted to $(2b-1)-(b-1) = b$, as illustrated in Figure 3.3. Now we argue that such neighbouring sets of b vertices of any two vertices differ in at least one vertex. Observe that in G', B_1 and B_2 include complete graphs K_b, and the edges between B_1 and B_2 form a perfect matching. This is because the neighbouring sets of any two vertices differ by at least one vertex in G. Since the number of common neighbours between any two vertices is no more than $b-1$, G' is also $K_{2,b}$-free. □

Now we present the following lower bound for $a = b = 3$, i.e $R(K_{3,3}, K_{3,3})$.

Theorem 21. $R(K_{3,3}, K_{3,3}) > 11$.

Proof By Counter Example To establish the bound, we show graphs with n = 11 free from $K_{3,3}$ in 3.6 . □

Table 3.1: Lower bounds for $R(K_{a,b}, K_{a,b})$ from Inequality 3.1(left), Theorem 22 (middle) and Theorem 24 (right)

b / a	3	4	5	6	7	8	14	15	16
1	2,3,3	2,3,4	3,4,5	3,5,6	3,5,7	3,6,8	5,10,17	5,11,18	6,12,19
2	3,4,4	3,5,6	4,6,7	5,7,9	5,8,10	6,9,12	9,17,23	10,18,24	10, 19, 26
3	4,5,6	5,7,8	6,8,9	7,10,12	8,12,14	9,14,16	16,26,32	17,29,35	18,31,37
4		6,9,10	8,11,12	10,14,15	12,16,18	14,19,22	26,41,46	28,45,50	30,49,55
5			11,14,16	13,18,20	16,22,24	19,27,29	40,60,65	43,67,72	47,74,80
6				17,23,25	21,29,31	26,35,38	59,87,93	66,98,104	72,109,116
7					27,37,39	34,46,48	86,123,129	96,139,147	106,156,165
8						43,58,61	119,168,178	136,193,204	152,219,232
14							556,755,820	678,922,1005	817,1113,1219
15								836,1136,1246	1019,1385,1525
16									1254,1704,1886

3.4 Probabilistic lower bounds for $R(K_{a,b}, K_{a,b})$

Probabilistic analysis for existence of structures was devised by Paul Erdós for lower bounding the original Ramsey numbers. He suggested that to get a lower bound for the Ramsey number $R(a,b)$, we want some (ideally as large as possible) n so that we can somehow colour the edges of K_n using two colors (say red and blue) in such a way that we get neither a red K_a or a blue K_b. He emphasized that we don't actually need to see an example of such a colouring, we just need to know that one exists. He then developed a non-constructive method of choosing a very large n satisfying the constraints by a method that is now referred to as the Probabilistic method(refer to [6] for the exact analysis of Erdős).

In the first Section 3.4.1 we use the probabilistic method to prove lower bounds on $R(K_{a,b}, K_{a,b})$ that is a improvement over existing results. In the Section 3.4.2, we demonstrate even more improved lower bounds using the Lovász' local lemma.

3.4.1 Application of the probabilistic method

The best known lower bound on $R(K_{a,b}, K_{a,b})$ due to Chung and Graham [4] is

$$R(K_{a,b}, K_{a,b}) > \left(2\pi\sqrt{ab}\right)^{\left(\frac{1}{a+b}\right)} \cdot \left(\frac{a+b}{e^2}\right) \cdot 2^{\frac{ab-1}{a+b}} \tag{3.1}$$

We derive a tighter lower bound using the probabilistic method as follows.

Theorem 22. *For natural numbers a and b,*

$$R(K_{a,b}, K_{a,b}) > 2^{\left(\frac{ab-1}{a+b}\right)} \tag{3.2}$$

Proof Let n be the number of vertices of graph G. Then the total number of distinct $K_{a,b}$ possible is

$$\binom{n}{a} \cdot \binom{n-a}{b}$$

Each $K_{a,b}$ has exactly ab edges. Each edge can be either of color 1 or color 2 with equal probability. So probability of a particular $K_{a,b}$ of color 1 is $\left(\frac{1}{2}\right)^{ab}$. So probability that a particular $K_{a,b}$ of either color 1 or color 2 exists is

$$2 \cdot \left(\frac{1}{2}\right)^{ab} = 2^{1-ab} \tag{3.3}$$

So probalility p of any monochromatic $K_{a,b} =$

$$\binom{n}{a} \cdot \binom{n-a}{b} \cdot 2^{1-ab}. \tag{3.4}$$

Our objective is to choose as large n as possible with $p < 1$. So choosing $n = 2^{\left(\frac{ab-1}{a+b}\right)}$, we get,

$$\begin{aligned}
p &= \binom{n}{a} \cdot \binom{n-a}{b} \cdot 2^{1-ab} \\
&\leq \frac{n^a}{a!} \cdot \frac{(n-a)^b}{b!} \cdot 2^{1-ab} \\
&< n^a \cdot n^b \cdot 2^{1-ab} \\
&= n^{(a+b)} \cdot 2^{1-ab} \\
&= 2^{\left(\frac{ab-1}{a+b}\right)\cdot(a+b)} \cdot 2^{1-ab} \\
&= 1.
\end{aligned}$$

As the probability p is strictly less than 1, it guarantees our requirement and thus gives a lower bound. □

Improvement In the earlier proof of lower bound, we neglected $a!$ and $b!$ from the denominator. If however we consider those terms we get a bound that is slightly better than the earlier bound. The Stirling's approximation gives a lower bound on $a!$.

$$a! \geq \sqrt{2\pi}\frac{a^{a+\frac{1}{2}}}{e^a} \text{ (Stirling's approximation)} \tag{3.5}$$

Theorem 23. $R(K_{a,b}, K_{a,b}) > \frac{(2\pi)^{\left(\frac{1}{a+b}\right)} \cdot a^{\left(\frac{a+\frac{1}{2}}{a+b}\right)} \cdot b^{\left(\frac{b+\frac{1}{2}}{a+b}\right)}}{e} \cdot 2^{\left(\frac{ab-1}{a+b}\right)}$

Proof Rewriting the expression for p from 3.4,

$$\begin{aligned}
p &= \binom{n}{a} \cdot \binom{n-a}{b} \cdot 2^{1-ab} \\
&\leq \frac{n^a}{a!} \cdot \frac{(n-a)^b}{b!} \cdot 2^{1-ab} \\
&< \frac{n^a}{a!} \cdot \frac{n^b}{b!} \cdot 2^{1-ab} \\
&< \frac{n^{a+b}}{a!b!} \cdot 2^{1-ab}
\end{aligned}$$

Now choosing $n = \frac{2\pi^{\left(\frac{1}{a+b}\right)} \cdot a^{\left(\frac{a+\frac{1}{2}}{a+b}\right)} \cdot b^{\left(\frac{b+\frac{1}{2}}{a+b}\right)}}{e} \cdot 2^{\left(\frac{ab-1}{a+b}\right)}$ and replacing $a!$ by $\sqrt{2\pi}\frac{a^{a+\frac{1}{2}}}{e^a}$ and

$b!$ by $\sqrt{2\pi}\frac{b^{b+\frac{1}{2}}}{e^b}$ as per 3.5, we get,

$$
\begin{aligned}
p &< \frac{n^{a+b}}{a!b!} \cdot 2^{1-ab} \\
&\le \frac{n^{a+b}}{\left(\sqrt{2\pi}\frac{a^{a+\frac{1}{2}}}{e^a}\right) \cdot \left(\sqrt{2\pi}\frac{b^{b+\frac{1}{2}}}{e^b}\right)} \cdot 2^{1-ab} \\
&= \frac{\left(\frac{2\pi^{\left(\frac{1}{a+b}\right)} \cdot a^{\left(\frac{a+\frac{1}{2}}{a+b}\right)} \cdot b^{\left(\frac{b+\frac{1}{2}}{a+b}\right)}}{e} \cdot 2^{\left(\frac{ab-1}{a+b}\right)}\right)^{a+b}}{\left(\sqrt{2\pi}\frac{a^{a+\frac{1}{2}}}{e^a}\right) \cdot \left(\sqrt{2\pi}\frac{b^{b+\frac{1}{2}}}{e^b}\right)} \cdot 2^{1-ab} \\
&= 1.
\end{aligned}
\tag{3.6}
$$

As the probability p is strictly less than 1, it guarantees our requirement and thus gives a lower bound. □

See Table 3.1 for the first two lower bounds for $R(K_{a,b}, K_{a,b})$ for each pair (a,b), due to Inequality 3.1 and Theorem 23, respectively. If we take the ratio of our lower bound and Graham's lower bound(say x), from Theorem 23 by Inequality 3.1, we get

$$
\begin{aligned}
x &= \frac{\frac{2\pi^{\left(\frac{1}{a+b}\right)} \cdot a^{\left(\frac{a+\frac{1}{2}}{a+b}\right)} \cdot b^{\left(\frac{b+\frac{1}{2}}{a+b}\right)}}{e} \cdot 2^{\left(\frac{ab-1}{a+b}\right)}}{\frac{2\pi^{\left(\frac{1}{a+b}\right)} \cdot a^{\left(\frac{1}{2\cdot a+b}\right)} \cdot b^{\left(\frac{1}{2\cdot(a+b)}\right)}}{e^2} \cdot (a+b) \cdot 2^{\left(\frac{ab-1}{a+b}\right)}} \\
=> x &= \frac{a^{\left(\frac{a}{a+b}\right)} \cdot b^{\left(\frac{b}{a+b}\right)}}{a+b} \cdot e
\end{aligned}
\tag{3.7}
$$

When $a = b$, from 3.7, we get

$$x = \frac{a}{2a} \cdot e = \frac{e}{2} \approx 1.359. \tag{3.8}$$

When $a << b$, as $a + b \approx b$, from 3.7, we get

$$x = \frac{b}{b} \cdot e \approx e. \tag{3.9}$$

So our lower bound gives an improvement that varies between 1.35 to Euler Constant e depending upon the values of a and b.

3.4.2 A lower bound for $R(K_{a,b},K_{a,b})$ using Lovász' local lemma

We are interested in the question of existence of a monochromatic $K_{a,b}$ in any bicolouring of the edges of K_n. Since the same edge may be present in many distinct $K_{a,b}$'s, the colouring of any particular edge may effect the monochromaticity in many $K_{a,b}$'s. This gives the motivation of use of Lovász' local lemma (see [11]) in this context.

Definition 2. *A directed dependency graph is a graph $G(V,E)$ where vertices are the events of a probability space. A directed edge between two vertices v_1 and v_2 indicates v_2 is dependent on v_1.*

Lemma 3. *Lovász Local Lemma*
Let $G(V,E)$ be a dependency graph for events $E_1,\ldots E_n$ in a probability space. Suppose that there exists $x_i \in [0,1]$ for $1 \le i \le n$ such that $Pr[E_i] \le x_i \prod_{\{i,j\}\in E}(1-x_j)$ then $Pr\left[\bigcap_{i=1}^{n} \overline{E_i}\right] \ge \prod_{i=1}^{n}(1-x_j)$.

A direct corollary of the lemma states

Corollary 1. *If every event E_i, $1 \le i \le m$ is dependent on at most d other events and $Pr[E_i] \le p$, and if $ep(d+1) \le 1$, then $Pr\left[\bigcap_{i=1}^{n} \overline{E_i}\right] > 0$.*

Theorem 24. *If $e \cdot 2^{1-ab} \cdot \left(ab\binom{n-2}{a+b-2}\binom{a+b-2}{b-1}+1\right) \le 1$, $R(K_{a,b},K_{a,b}) > n$*

Proof We consider a random bicolouring of the complete graph K_n in which each edge is independently coloured red or blue with equal probability. Let S be the set of edges of an arbitrary $K_{a,b}$, and let E_S be the event that all edges in this $K_{a,b}$ are coloured monochromatically. For each such S, the probability of E_S is $P(E_S) = 2^{1-ab}$. We enumerate the sets of edges of all possible $K_{a,b}$'s as S_1,S_2,...,S_m, where $m = \binom{n}{a}\binom{n-a}{b}$. Clearly, each event E_{S_i} is mutually independent of all the events E_{S_j} from the set

$$\{E_{S_j} : |S_i \cap S_j| = 0\} \tag{3.10}$$

since for any such S_j , S_i and S_j share no edges. For each E_{S_i}, the number of events outside this set satisfies the inequality

$$|\{E_{S_j} : |S_i \cap S_j| \geq 1\}| \leq ab\binom{n-2}{a+b-2}\binom{a+b-2}{b-1} \tag{3.11}$$

as every S_j in this set shares at least one edge with S_i, and therefore such an S_j shares at least two vertices with S_i. We can choose the rest of the $a+b-2$ vertices of S_j from the remaining $n-2$ vertices of K_n, out of which we can choose $b-1$ for one partite of S_j, and the remaining $a-1$ to form the second partite of S_j, yielding a $K_{a,b}$ that shares at least one edge with S_i. We apply Corollary 1 to the set of events $E_{S_1}, E_{S_2}, ..., E_{S_m}$, with

$$p = 2^{1-ab}\,,\, d = ab\binom{n-2}{a+b-2}\binom{a+b-2}{b-1}\,, \tag{3.12}$$

yields

$$e \cdot 2^{1-ab} \cdot \left(ab\binom{n}{a+b-2}\binom{a+b-2}{b-1} + 1\right) \leq 1 => Pr\left[\bigcap_{i=1}^{m} \overline{E}_{S_i}\right] > 0 \tag{3.13}$$

This non-zero probability (of none of the events E_{S_i} occuring, for $1 \leq i \leq m$) implies the existence of some bicolouring of the edges of K_n with no monochromatic $K_{a,b}$, thereby establishing the theorem. $\square$

Solving the inequality in the statement of Theorem 24, we can compute lower bounds for $R(K_{a,b}, K_{a,b})$, for natural numbers a and b. Such lower bounds for some larger values of a and b show significant improvements over the bounds computed using Theorem 22 (see Table 3.1).

Chapter 4

The unbalanced off-diagonal case: $R(K_{a,b}, K_{c,d})$

$R(K_{a,b}, K_{c,d})$ is the minimum number n so that any n-vertex simple undirected graph G must contain a $K_{a,b}$ or its complement G' must contain the complete bipartite graph $K_{c,d}$. Equivalently, $R(K_{a,b}, K_{c,d})$ is the minimum number n such that any 2-coloring of the edges of an n-vertex complete undirected graph would contain a monochromatic $K_{a,b}$ or a monochromatic $K_{c,d}$.

4.1 A constructive lower bound for $R(K_{2,b}, K_{2,d})$

Now we present a constructive lower bound as follows by designing an explicit construction.

Theorem 25. *$R(K_{2,b}, K_{2,d}) > b+d+1$, for all integers $d \geq b \geq 2$.*

Proof For $d \geq b \geq 2$, we demonstrate the existence of a $K_{2,b}$-free graph with $b+d+1$ vertices, such that its complement graph does not contain any $K_{2,d}$. The construction is illustrated for specific values of b and d in Figure 4.1. We have the following three exhaustive cases.

Case 1:

If $b = 2m$ for an integer m, then arrange all the vertices around a circle, numbering vertices as $v_0, v_1, v_2, ..., v_{b+d}$, and connect each vertex to its m nearest neighbours in

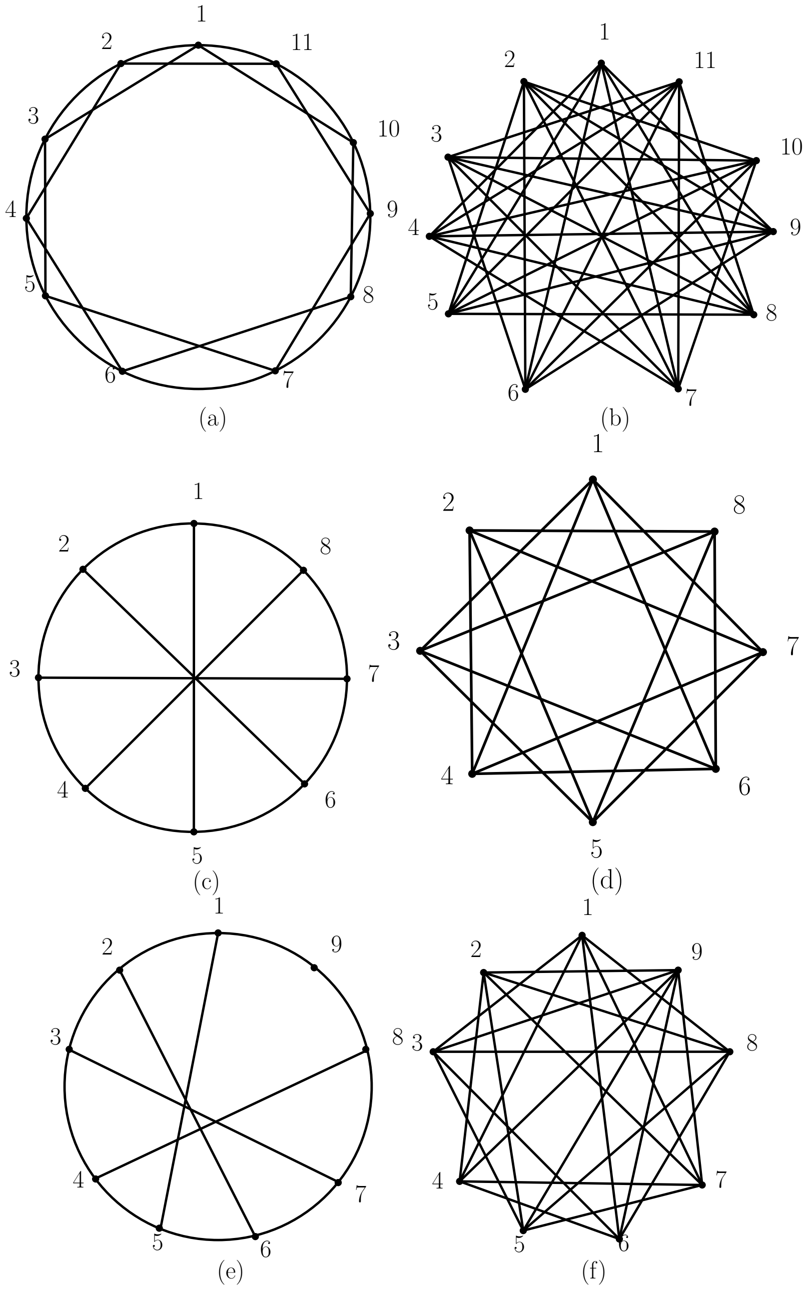

Figure 4.1: (i) $R(K_{2,4}, K_{2,6}) > 11$: (a) graph G_1 is $K_{2,4}$-free, and and (b) graph G_1' is $K_{2,6}$-free, (ii) $R(K_{2,3}, K_{2,4}) > 8$: (c) graph G_2 is $K_{2,3}$-free, and and (d) graph G_2' is $K_{2,4}$-free, (iii) $R(K_{2,3}, K_{2,5}) > 9$: (e) graph G_3 is $K_{2,3}$-free, and (f) graph G_3' is $K_{2,5}$-free.

clockwise (as well as counterclockwise) directions along the circle. See graph G_1 in Figure 4.1(a) for an example with $b = 4$ and $d = 6$. Observe that the constructed graph G is b-regular, and its complement graph is therefore d-regular. We claim that G does not have a $K_{2,b}$ since no two vertices in G share more than $b-2$ neighbours.

We first show that for all i, $0 \leq i \leq b+d$, the vertex v_i shares exactly $2(m-1) = b-2$ neighbours with v_{i+1}. Here and henceforth, all arithmetic operations on indices of vertices are modulo $b+d+1$. There are exactly $m-1$ neighbours common to v_i and v_{i+1} in the clockwise (respectively, counterclockwise) direction of v_i (v_{i+1}), resulting in a total of $2(m-1)$ common vertices. Similarly, the number of vertices shared by v_i with its neighbouring clockwise vertex v_{i-1} is also $b-2$. Now consider the remaining counterclockwise neighbours v_{i+k} of v_i in G, $2 \leq k \leq m$. Observe that vertices v_i and v_{i+k} share exactly $2(m-k)+(k-1) = 2m-k-1 = b-k-1$ neighbours; $m-k$ vertices clockwise (respectively, counterclockwise) of v_i (respectively, v_{i+k}), and $k-1$ vertices clockwise of v_{i+1} and counterclockwise of v_i. So, the total number of shared neighbours between v_i and v_{i+k} (and symmetrically, between v_i and v_{i-k}), is certainly no more than $2(m-1) = b-2$. For the d non-adjacent vertices v_j of v_i, clearly v_j and v_i do not share more than $m < b-2$ common neighbours. This implies that the graph G is $K_{2,b}$-free.

Now consider the complement graph G' of G. Since we have $b+d+1$ vertices, the complement graph G' is d-regular if and only if the graph G is b-regular. See Figure 4.1(b) for the complement graph G'_1 of G_1, for $b = 4$ and $d = 6$. The complement graph G' can have a $K_{2,d}$ only if two vertices share all their neighbours. Each pair of vertices differ in at least two vertices in their neighbourhood in G, since any pair of two vertices can share at most $b-2$ vertices in the b-regular graph G. This ensures that no two vertices can have all neighbours common in G'. For any vertex pair (v_i,v_j), even if the neighbourhood of v_i includes v_j, v_i still has some neighbour v_k that is not a neighbour of v_j in G, and (similarly) v_j has some neighbour v_l that is not a neighbour of v_i in G. In G' therefore, v_k is a neighbour of v_j but not a neighbour of v_i, and v_l is a neighbour of v_i but a neighbour of v_j. Therefore, G' is $K_{2,d}$-free.

Case 2:

If $b = 2m+1$ for an integer m, and $b+d+1$ is even (i.e., d is even), then arrange and name the vertices around a circle as in Case 1, and connect each vertex to its m nearest neighbours in counterclockwise as well as clockwise directions around the circle. Also, connect each vertex v_i to the vertex $v_{i+\frac{b+d+1}{2}}$, directly opposite to it on the circle; note that no two vertices share such a common directly opposite neighbour. The resulting graph G is b-regular. As shown in Case 1, this graph G does not have any $K_{2,b}$ as no two vertices share more than $2(m-1) = b-3 < b-2$ common neighbours. The complement graph G' is again d-regular, as in Case 1. The construction is illustrated for the case of $R(K_{2,3}, K_{2,4})$ in Figure 4.1(c) and (d). The only way G' can have a $K_{2,d}$ is if two vertices share all their neighbours in G'. Since two vertices G share less than $b-2$ vertices in G, they cannot have all neighbours common in G'. This can be shown in a manner similar to that in Case 1. So, G' is $K_{2,d}$-free.

Case 3:

If $b = 2m+1$ for some integer m, and $b+d+1$ is odd (i.e., d is odd), then arrange and name the vertices around a circle as in Cases 1 and 2, and connect (i) each vertex to its m nearest neighbours in counterclockwise as well as clockwise directions, and (ii) connect each vertex v_i to vertex $v_{i+\lfloor\frac{b+d+1}{2}\rfloor}$, for all i, $1 \leq i \leq \lfloor\frac{b+d+1}{2}\rfloor - 1$. This results in a graph G with $b+d$ vertices of degree b and one vertex v_{b+d} of degree $b-1$. Observe that as in Cases 1 and 2, the number of common neighbours for any two vertices in G is no more than $2(m-1) = b-3 < b-2$. This graph G is therefore free from any $K_{2,b}$.

We now show that G' is $K_{2,d}$-free. Observe that every vertex of the complement graph G' has degree d, except v_{b+d} whose degree is $d+1$. The construction is illustrated for the case of $R(K_{2,3}, K_{2,5})$ in Figure 4.1(e) and (f). The only way G' can have a $K_{2,d}$ is (i) if some d-degree vertex shares all its neighbours with some other d-degree vertex in G' (as in Cases 1 and 2), or (ii) if any d of the $d+1$ neighbours of the $d+1$-degree vertex v_{b+d}, are shared with a d-degree vertex in G'. Two d-degree vertices disagreeing on at least two neighbours cannot yield a $K_{2,d}$, as seen in Cases 1 and 2. So, we need to consider only the later case involving vertex v_{b+d}, whose degree is $d+1$ in G'. Consider a d-degree vertex v_i of G' and the vertex v_{b+d}. Since these two vertices share at most $b-2$ vertices in G, there is at least one neighbouring vertex v_j of v_{b+d} in G, that is not a common neighbour in G for v_i and v_{b+d}. So, v_j not connected to v_i in G and therefore a v_j is a neighbour of v_i in G'. Also, v_j is connected to v_{b+d} in G and therefore not a neighbour of v_{b+d} in G'. So, G' does not have a $K_{2,d}$ where v_i and v_{b+d} should share d neighbours.

$\square$

Now we derive a lower bound on such numbers using probabilistic method.

Theorem 26. *For all $n \in N$ and $0 < p < 1$, if $\binom{n}{a}\binom{n-a}{b}p^{ab} + \binom{n}{c}\binom{n-c}{d}(1-p)^{cd} < 1$, then $R(K_{a,b}, K_{c,d}) > n$.*

Proof To proof the above theorem, Lets consider some random bi-colouring Of the complete graph K_n with colours red and blue with probabilities p and $(1-p)$ respectively. Then the probability that a particular red $K_{a,b}$ exists is p^{ab}, hence the probability that some red $K_{a,b}$ exists is $\binom{n}{a}\binom{n-a}{b}p^{ab}$. Similarly the probability that a particular blue $K_{c,d}$ exists is $(1-p)^{cd}$, hence the probability that some blue $K_{c,d}$ exists is $\binom{n}{c}\binom{n-c}{d}(1-p)^{cd}$. So the probability that the bicoloured K_n contains any red $K_{a,b}$ or any blue $K_{c,d}$ is $\binom{n}{a}\binom{n-a}{b}p^{ab} + \binom{n}{c}\binom{n-c}{d}(1-p)^{cd}$, which is assumed to be less than 1 in the premise of Theorem 26. Hence the probability of the complementary event, i.e. there is neither any red $K_{a,b}$ or any blue $K_{c,d}$, is non zero, hence there exist some 2-colouring for which there is neither any red $K_{a,b}$ or any blue $K_{c,d}$ as subgraph. □

4.2 A lower bound for $R(K_{a,b}, K_{c,d})$ using Lovász' local lemma

We are interested in the question of existence of a monochromatic $K_{a,b}$ or a monochromatic $K_{c,d}$ in any bicolouring of the edges of K_n. Since the same edge may be shared by many distinct $K_{a,b}$'s and $K_{c,d}$'s, the colouring of any particular edge may effect the monochromaticity in many $K_{a,b}$'s and $K_{c,d}$'s. This gives the motivation of use of Lovász' local lemma 3 in this context.

Theorem 27. *If for some $0 < p < 1$, $\left\{ ab\binom{n-2}{a-1}\binom{n-a-1}{b-1} + 1 \right\} p^{ab} e^{1+\frac{ab}{cd}} \leq 1$ and $\left\{ cd\binom{n-2}{c-1}\binom{n-c-1}{c-1} + 1 \right\} e^{-pcd} e^{1+\frac{cd}{ab}} \leq 1$, then $R(K_{a,b}, K_{c,d}) > n$.*

Proof We consider a random bicolouring of the complete graph K_n in which each edge is independently coloured red or blue with probabilities p and $(1-p)$ respectively. Let S be the set of edges of an arbitrary $K_{a,b}$,T be the set of edges of an arbitrary $K_{c,d}$, . let E_S be the event that all edges in the $K_{a,b}$ S are coloured monochromatically red and let E_T be the event that all edges in the $K_{c,d}$ T are coloured monochromatically blue. For each such S, the probability of E_S is $P(E_S) = p^{ab}$. Similarly For each such T, the

Table 4.1: Lower bounds for $R(K_{a,b}, K_{c,d})$ from Theorem 26 (left) and Theorem 27 (right)

c,d / a,b	4,5	4,6	5,6	5,7	6,7	6,8	12,13	12,14	13,14	14,15	15,16
3,4	9,9	9,10	10,12	11,14	12,16	13,18	24,79	25,89	26,102	28,133	30,173
3,5	12,9	12,10	12,12	12,14	12,17	13,19	24,87	25,99	26,114	28,149	30,196
3,6	14,11	14,12	14,13	14,14	14,17	14,19	24,93	25,106	26,122	28,162	30,213
3,7	15,13	16,14	16,14	16,15	16,17	16,20	24,97	25,111	26,129	28,171	30,227
4,5	14,12	15,12	15,13	15,15	15,17	15,20	24,96	25,109	26,127	28,168	30,222
4,6	15,12	17,15	18,15	18,15	18,17	18,20	24,101	25,115	26,134	28,178	30,237
4,7	15,17	18,18	21,18	22,18	22,18	22,20	24,104	25,119	26,139	28, 186	30,248
5,6	15,13	18,15	22,19	23,19	23,20	23,21	24,106	25,121	26,142	28,189	30,253
5,7	15,15	18,15	23,19	27,23	28,24	28,24	28,109	28,125	28,146	28,196	30,262
10,11	20,144	20,150	23,157	28,161	36,166	43,168	188,182	188,183	188,183	188,226	188,306
10,12	21,173	21,181	23,190	28,195	36,200	43,204	226,221	226,221	226,222	226,227	226,307
10,13	22,205	22,215	23,226	28,232	36,239	43,243	269,265	269,265	269,266	269,267	269,309
11,12	22,215	22,226	23,237	28,244	36,251	43,256	283,279	283,280	283,281	283,282	283,309
11,13	23,258	23,272	23,286	28,295	36,303	43,309	341,338	341,339	341,340	341,342	341,343
12,13	24,320	24,338	24,357	28,368	36,380	43,388	415,426	426,427	426,429	426,431	426,432
13,14	26,476	26,504	26,535	28,554	36,573	43,585	426,648	513,650	623,652	639,656	639,659
14,15	28,704	28,750	28,799	28,829	36,859	43,879	426,982	513,985	639,989	935,994	957,999
15,16	30,1038	30,1111	30,1189	30,1236	36,1285	43,1317	426,1482	513,1488	639,1493	957,1502	1399,1509

probability of E_T is $P(E_T) = (1-p)^{cd}$. We enumerate the sets of edges of all possible $K_{a,b}$'s and $K_{c,d}$'s as $A_1, A_2, \ldots, A_m$, where $m = \binom{n}{a}\binom{n-a}{b} + \binom{n}{c}\binom{n-c}{d}$. Clearly, each event E_{A_i} is mutually independent of all the events E_{A_j} from the set $\{E_{A_j} : |A_i \cap A_j| = 0\}$; since for any such A_j , A_i and A_j share no edges. Now again as the events can be a monochromatic $K_{a,b}$ or $K_{c,d}$, Let A_{ab} denote a $K_{a,b}$ and A_{cd} denote a $K_{c,d}$.

For each $E_{A_{ab}}$, the number of events outside this set satisfies the inequality $|\{E_{A_j} : |A_{ab} \cap A_j| \geq 1\}| \leq ab\{\binom{n-2}{a-1}\binom{n-a-1}{b-1} + \binom{n-2}{c-1}\binom{n-c-1}{d-1}\}$; every A_j in this set shares at least one edge with A_{ab}, and therefore such an A_j shares at least two vertices with A_{ab}. If this A_j is a $K_{a,b}$, then We can choose the rest of the $a+b-2$ vertices of A_j from the remaining $n-2$ vertices of K_n, out of which we can choose $a-1$ for one partite of A_j, and the remaining $b-1$ to form the second partite of A_j, yielding a $K_{a,b}$ that shares at least one edge with A_{ab}. On the other hand, if this A_j is a $K_{c,d}$, then We can choose the rest of the $c+d-2$ vertices of A_j from the remaining $n-2$ vertices of K_n, out of which we can choose $c-1$ for one partite of A_j, and the remaining $d-1$ to form the second partite of A_j, yielding a $K_{a,b}$ that shares at least one edge with A_{ab}. Similarly, For each $E_{A_{cd}}$, the number of events that shares atleast one edge satisfies the inequality $|\{E_{A_j} : |A_{cd} \cap A_j| \geq 1\}| \leq cd\{\binom{n-2}{a-1}\binom{n-a-1}{b-1} + \binom{n-2}{c-1}\binom{n-c-1}{d-1}\}$. By applying Theorem 3, we want to show that

$$Pr\left[\bigcap_{i=1}^{m} \overline{E}_{A_i}\right] > 0. \tag{4.1}$$

This non-zero probability (of none of the events E_{A_i} occuring, for $1 \le i \le m$) implies the existence of some bicolouring of the edges of K_n with no red $K_{a,b}$ or blue $K_{c,d}$, thereby establishing the theorem. The Inequality 4.1 is satisfied if the following conditions hold.

$$\begin{aligned} Pr[E_{A_{ab}}] &\le x_{ab}\,(1-x_{ab})^{ab\binom{n-2}{a-1}\binom{n-a-1}{b-1}}\,(1-x_{cd})^{ab\binom{n-2}{c-1}\binom{n-c-1}{d-1}} \\ Pr[E_{A_{cd}}] &\le x_{cd}\,(1-x_{ab})^{cd\binom{n-2}{a-1}\binom{n-a-1}{b-1}}\,(1-x_{cd})^{cd\binom{n-2}{c-1}\binom{n-c-1}{d-1}}, \\ &\qquad\qquad \text{for some } x_{ab}, x_{cd}. \end{aligned} \tag{4.2}$$

Choosing $x_{ab} = \frac{1}{ab\binom{n-2}{a-1}\binom{n-a-1}{b-1}+1}$, $x_{cd} = \frac{1}{cd\binom{n-2}{c-1}\binom{n-c-1}{d-1}+1}$ and using the inequalities $(1-p)^{cd} \le e^{-pcd}$ and $\left(1-\frac{1}{d+1}\right)^{d} \ge e$, we get

$$\begin{aligned} &\left\{ab\binom{n-2}{a-1}\binom{n-a-1}{b-1}+1\right\} p^{ab} e^{1+\frac{ab}{cd}} \le 1, \text{ and} \\ &\left\{cd\binom{n-2}{c-1}\binom{n-c-1}{c-1}+1\right\} e^{-pcd} e^{1+\frac{cd}{ab}} \le 1. \end{aligned} \tag{4.3}$$

To get a lower bound on $R(K_{a,b},K_{c,d})$, we choose the largest value of n, such that both of these conditions are satisfied. □

Solving the inequality in the statement of Theorem 27, we can compute lower bounds for $R(K_{a,b},K_{a,b})$, for natural numbers a and b. Such lower bounds for some larger values of a and b show significant improvements over the bounds computed using Theorem 26 (see Table 4.2).

4.3 The balanced off-diagonal case: $R(K_{a,a},K_{b,b})$

$R(K_{a,a},K_{b,b})$ is the minimum number n so that any n-vertex simple undirected graph G must contain a $K_{a,a}$ or its complement G' must contain the complete bipartite graph $K_{b,b}$. Equivalently, $R(K_{a,a},K_{b,b})$ is the minimum number n such that any 2-coloring of the edges of an n-vertex complete undirected graph would contain a monochromatic

$K_{a,a}$ or a monochromatic $K_{b,b}$. As this case is equivalent to the unbalanced off-diagonal case with proper substitutions, we exclude the extended analysis of the lower bounds which are as follows.

Corollary 2. *For all* $n \in N$ *and* $0 < p < 1$, *if* $\binom{n}{a}\binom{n-a}{a}p^{a^2} + \binom{n}{b}\binom{n-b}{b}(1-p)^{b^2} < 1$, *then* $R(K_{a,a}, K_{b,b}) > n$.

Corollary 3. *If for some* $0 < p < 1$, $\left\{a^2\binom{n-2}{a-1}\binom{n-a-1}{a-1}+1\right\}p^{a^2}e^{1+\frac{a^2}{b^2}} \leq 1$ *and* $\left\{b^2\binom{n-2}{b-1}\binom{n-b-1}{b-1}+1\right\}e^{-pb^2}e^{1+\frac{b^2}{a^2}} \leq 1$, *then* $R(K_{a,a}, K_{b,b}) > n$.

Chapter 5

Upper Bounds on $R(K_{a,b}, K_{a,b})$ and $R(K_{a,b}, K_{c,d})$

5.1 Existence and General Upper Bound

As we have already seen, in many cases we are able to establish a lower bound for some Bipartite Ramsey number $R(K_{a,b}, K_{a,b})$ by finding a complete graph K_n and a 2-coloring of K_n such that there is no monochromatic $K_{a,b}$. Bipartite Ramsey theorem states that such a number exists for all a and b. Existence proof is achieved by proving a explicit bound. We show the Bipartite Ramsey number $R(K_{a,b}, K_{a,b})$ is bounded by Ramsey number $R(K_{a+b}, K_{a+b})$.

Theorem 28. $R(K_{a,b}, K_{a,b}) \leq R(K_{a+b}, K_{a+b})$.

From Ramsey theorem we know that for any positive integers a and b, $R(K_a, K_b)$ always exist. Hence $R(K_{a+b}, K_{a+b})$ also exists. $R(K_{a+b}, K_{a+b})$ is the minimum number such that any bicoloring of the graph with this number of vertices always contain a monochromatic K_{a+b}. As K_{a+b} always contains a subgraph $K_{a,b}$, hence the number that guarantees a monochomatic K_{a+b} always guarantees a monochomatic $K_{a,b}$ and hence

$$R(K_{a,b}, K_{a,b}) \leq R(K_{a+b}, K_{a+b}) \tag{5.1}$$

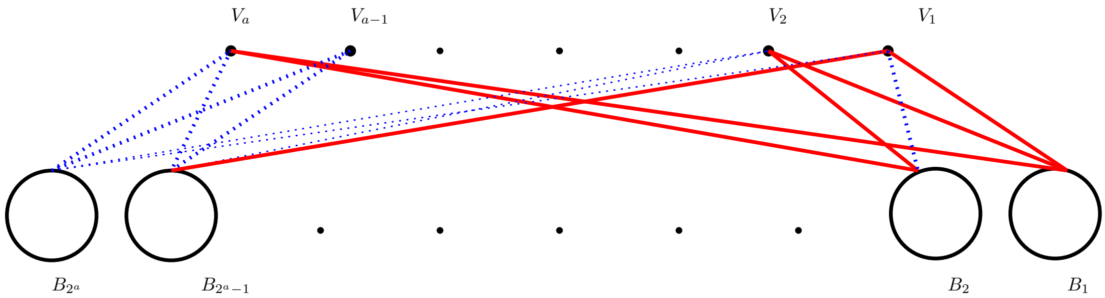

Figure 5.1: Construction for the upper bound.

5.2 A Bad Upper Bound on $R(K_{a,b}, K_{a,b})$

Theorem 29. *$R(K_{a,b}, K_{a,b}) \leq 2^a \cdot R(K_{a-1,b}, K_{a-1,b})$, $a < b$.*

Proof We take $2^a \cdot R(K_{a-1,b}, K_{a-1,b})$ number of vertices and show that $R(K_{a,b}, K_{a,b})$ is bounded by it. 1). We take out exactly a vertices, name them as $v_1, v_2, ..., v_a$. 2). Now color the entire graph randomly using two colors say red and blue. 3). Then we split the vertices of the graph into exactly 2^a groups, $B_1, B_2, ..., B_{2^a}$ based on the color of the edges they are connected to from $v_1, v_2, ..., v_a$, i.e if a vertex connected to all vertices of $v_1, v_2, ..., v_a$ by edges colored red, then it is placed in group B_1 and if a vertex connected to all vertices of $v_1, v_2, ..., v_a$ by edges colored blue, then it is placed in group B_{2^a}. The construction is illustrated in 5.1. Now we claim that $\exists B_i \in \{B_1, B_2, ..., B_{2^a}\}$, such that $|B_i| \geq R(K_{a-1,b}, K_{a-1,b})$.

This can be proved with a contradiction as follows:
if $\forall B_i \in \{B_1, B_2, ..., B_{2^a}\}, |B_i| < R(K_{a-1,b}, K_{a-1,b})$, then total number of vertices cannot exceed $2^a \cdot \{R(K_{a-1,b}, K_{a-1,b}) - 1\} + a$, which is less than the number of vertices we have.

Case 1:$B_i \in \{B_2, ..., B_{2^a-1}\}$, such that $|B_i| \geq R(K_{a-1,b}, K_{a-1,b})$.
Let B_i contains a red $K_{a-1,b}$. Then there exists atleast one vertex from $v_1, v_2, ..., v_a$ that is connected to every vertex of B_i with a edge colored red. This vertex combined with the $K_{a-1,b}$ forms the red $K_{a,b}$. Similar argument holds if B_i contains a blue $K_{a-1,b}$.

Case 2:$B_i \in \{B_1, B_{2^a}\}$, such that $|B_i| \geq R(K_{a-1,b}, K_{a-1,b})$.
As $v_1, v_2, ..., v_a$ are connected to B_1 by edges that are colored all red, if $|B_1| \geq R(K_{a-1,b}, K_{a-1,b})$, then as $R(K_{a-1,b}, K_{a-1,b}) > b$, the vertices $v_1, v_2, ..., v_a$ along with b vertices from $|B_1|$ form a red $K_{a,b}$. Again As $v_1, v_2, ..., v_a$ are connected to B_{2^a} by edges that are

colored all blue, if $|B_{2^a}| \geq R(K_{a-1,b}, K_{a-1,b})$, then as $R(K_{a-1,b}, K_{a-1,b}) > b$, the vertices $v_1, v_2, ..., v_a$ along with b vertices from $|B_{2^a}|$ form a red $K_{a,b}$. This completes our proof that

$$R(K_{a,b}, K_{a,b}) \leq 2^a \cdot R(K_{a-1,b}, K_{a-1,b}), \text{ for } a < b. \tag{5.2}$$

□

5.3 Comparision of the Upper Bounds

From Inequality 5.1 and Inequality 5.2, replacing b with a, we have two upper bounds on $R(K_{a,a}, K_{a,a})$, which are as follows:

$$R(K_{a,a}, K_{a,a}) \leq R(K_{2a}, K_{2a}). \tag{5.3}$$

$$R(K_{a,a}, K_{a,a}) \leq 2^a \cdot R(K_{a-1,a}, K_{a-1,a}). \tag{5.4}$$

Now as Ramsey number $R(a,a) < 2^{2a}$, from inequality 5.3, we get,

$$R(K_{a,a}, K_{a,a}) \leq 2^{4a}. \tag{5.5}$$

Expanding inequality 5.4, we get,

$$\begin{aligned}
R(K_{a,a}, K_{a,a}) &\leq 2^a \cdot R(K_{a-1,a}, K_{a-1,a}). \\
&\leq 2^{a+a-1} \cdot R(K_{a-2,a}, K_{a-2,a}) \\
&\leq 2^{a+a-1+...+2} \cdot R(K_{1,a}, K_{1,a}) \\
&\leq 2^{a+a-1+...+2} \cdot 2a \\
&= 2^{a+a-1+...+2+1} \cdot a \\
&= 2^{\frac{a \cdot (a-1)}{2}} \cdot a \\
=> R(K_{a,a}, K_{a,a}) &\leq 2^{\frac{a \cdot (a-1)}{2}} \cdot a
\end{aligned} \tag{5.6}$$

As $2^{4a} < 2^{\frac{a\cdot(a-1)}{2}} \cdot a$, inequality 5.5 gives a better upper bound for $R(K_{a,a},K_{a,a})$ than inequality 5.6, thus inequality 5.1 gives us a better bound.

5.4 Inductive upper bound for $R(K_{a,b},K_{c,d})$

Theorem 30. *For $a \geq 2$, $b \geq 2$,$c \geq 2$, $d \geq 2$, $R(K_{a,b},K_{c,d}) \leq R(K_{a,b-1},K_{c,d}) +$ $R(K_{a,b},K_{c,d-1})$.*

Proof We will proceed by induction on $a+b+c+d$. First we consider the base case in which $a+b+c+d = 8$. The only way this can be true is if $a = b = c = d = 2$, and it is known that $R(K_{2,2},K_{2,2}) = 6$ [3], where as $R(K_{1,2},K_{2,2}) = 4$ [12], hence this proves the base case. Now by induction hypothesis, we assume that the theorem holds whenever $a+b+c+d < N$, for some positive integer N. Now our objective is to show that theorem also holds for $a+b+c+d = N$. We assume that $a+b+c+d = N$. Then $a+b+c+d-1 < N$, so by induction hypothesis, $R(K_{a,b-1},K_{c,d})$ and $R(K_{a,b},K_{c,d-1})$ exist and are finite.

Consider a complete graph on $R(K_{a,b-1},K_{c,d})+R(K_{a,b},K_{c,d-1})$ vertices. Pick a vertex v from the graph, and partition the remaining vertices into two sets M and N, such that for every vertex w, w is in M if (v,w) pair is blue, and w is in N if (v,w) is red. As the graph has $R(K_{a,b-1},K_{c,d})+R(K_{a,b},K_{c,d-1}) = |M|+|N|+1$ vertices, by symmetry, it follows that either

$$|M| \geq R(K_{a,b-1},K_{c,d}) \text{ or } |N| \geq R(K_{a,b},K_{c,d-1}).$$

In the former case, if M has a red $K_{c,d}$ then so does the original graph and we are finished. Otherwise M has a blue $K_{a,b-1}$ and so $M \cup \{v\}$ has blue $K_{a,b}$ by definition of M. In the latter case, if N has a blue $K_{a,b}$ then so does the original graph and we are finished. Otherwise N has a red $K_{c,d-1}$ and so $N \cup \{v\}$ has blue $K_{c,d}$. □

Now using Theorem Theorem 30,

$$\begin{aligned}
R(K_{a,b},K_{c,d}) &\leq R(K_{a,b-1},K_{c,d})+R(K_{a,b},K_{c,d-1}) \\
&\leq R(K_{a-1,b-1},K_{c,d})+R(K_{a,b-1},K_{c,d-1})+ \\
&\quad R(K_{a,b-1},K_{c,d-1})+R(K_{a,b},K_{c-1,d-1}) \\
&\leq R(K_{a-1,b-1},K_{c,d})+2(R(K_{a-1,b-1},K_{c,d-1})+ \\
&\quad R(K_{a,b-1},K_{c-1,d-1}))+R(K_{a,b},K_{c-1,d-1})
\end{aligned}$$

$$
\begin{aligned}
&\leq R(K_{a-1,b-1},K_{c,d}) + 2R(K_{a-1,b-1},K_{c,d}) + \\
&\quad 2R(K_{a,b},K_{c-1,d-1}) + R(K_{a,b},K_{c-1,d-1}) \\
&\leq 3\left(R(K_{a-1,b-1},K_{c,d}) + R(K_{a,b},K_{c-1,d-1})\right) \\
\Rightarrow R(K_{a,b},K_{c,d}) &\leq 3\left(R(K_{a-1,b-1},K_{c,d}) + R(K_{a,b},K_{c-1,d-1})\right).
\end{aligned}
$$

5.5 Edge analysis for $K_{a,b}$ free structures

In order to establish bounds on $R(K_{a,b},K_{a,b})$, one strategy can be bounding the maximum possible edges that can be possible in a graph free from $K_{a,b}$. This is because using this bound we can get a graph G that is free from $K_{a,b}$ and then we can concentrate on the complement graph. Though this might not give the exact value of $R(K_{a,b},K_{a,b})$, this should give a tighter bounds. One such theorem that gives a bound on number of edges in a $K_{a,b}$ free graph is Kővári-Sós-Turán theorem that is discussed below.

Theorem 31 (Kővári-Sós-Turán Theorem). *Let $G(V,E)$ be a graph that is free from $K_{a,b}$ $(1 \leq a \leq b)$as a subgraph and $n = |V|$, then $|E| = O(n^{2-\frac{1}{a}})$*

Proof To prove the bound we count the number of star configurations(i.e a vertex that has a neighbours). Let C be all such configurations. Then

$$|C| = \sum_{i=1}^{n} \binom{d_i}{a} \leq (b-1)\binom{n}{a} \tag{5.7}$$

(we assume that each vertex has a degree greater than or equal to a.).

Holders inequality for sums: Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p,q \geq 1$.Then

$$\sum_{i=1}^{n} a_k \cdot b_k \leq \left(\sum_{i=1}^{n} a_k{}^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} b_k{}^q\right)^{\frac{1}{q}} \tag{5.8}$$

Now replacing p by a ,a_k by d_i,b_k by 1, $\frac{1}{q} = 1 - \frac{1}{a} = \frac{a-1}{a}$ and from 5.8 we have,

$$\begin{aligned}
\sum_{i=1}^{n} d_i \cdot 1 &\leq \left(\sum_{i=1}^{n} {d_i}^a\right)^{\frac{1}{a}} \cdot \left(\sum_{i=1}^{n} 1^{\left(\frac{a}{a-1}\right)}\right)^{\frac{a-1}{a}} \\
=> \left(\sum_{i=1}^{n} d_i\right)^a &\leq \left(\sum_{i=1}^{n} {d_i}^a\right) \cdot n^{a-1} \\
=> \left(\sum_{i=1}^{n} {d_i}^a\right) &\geq \frac{\left(\sum_{i=1}^{n} d_i\right)^a}{n^{a-1}} = \frac{(2|E|)^a}{n^{a-1}} \qquad (5.9)
\end{aligned}$$

Again as $\sum_{i=1}^{n} \binom{d_i}{a} = \Omega\left(\sum_{i=1}^{n} {d_i}^a\right)$, from 5.7 and 5.9 we get,

$$\begin{aligned}
(2|E|)^a &\leq c \cdot n^{a-1} \cdot (b-1)\binom{n}{a} \qquad \text{(c is some constant)} \\
&\leq c \cdot n^{a-1} \cdot (b-1) \cdot \frac{n^a}{a!} \\
=> |E| &\leq O(n^{2-\frac{1}{a}}) \qquad (5.10)
\end{aligned}$$

□

We may do the exact calculations to get exact bounds as follows.

$$\begin{aligned}
\sum_{i=1}^{n} \binom{d_i}{a} &\geq \sum_{i=1}^{n} \left(\frac{d_i}{a}\right)^a \\
&= \frac{1}{a^a} \cdot \sum_{i=1}^{n} {d_i}^a \qquad (5.11)
\end{aligned}$$

Now from 5.7 , 5.9 and 5.11 we get,

$$\begin{aligned}
(2|E|)^a &\leq a^a \cdot n^{a-1} \cdot (b-1)\binom{n}{a} \qquad \text{(c is some constant)} \\
&\leq a^a \cdot n^{a-1} \cdot (b-1) \cdot \frac{n^a}{a!} \\
=> |E| &\leq \frac{a}{2} \cdot \sqrt[a]{\frac{b-1}{a!}} \cdot n^{2-\frac{1}{a}} \qquad (5.12)
\end{aligned}$$

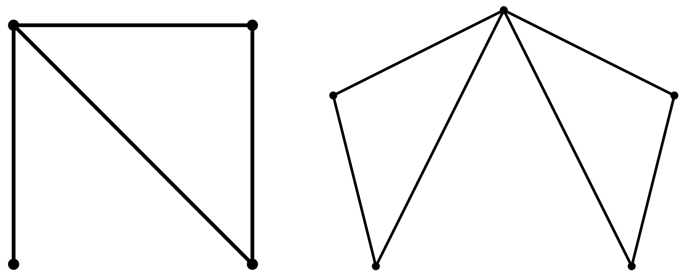

Figure 5.2: Addition of maximum edges with graph free from $K_{2,2}$ with $n = 4$ and $n = 5$.

Now We can analyse value of this $|E|$ with the actual values for small values of n, a and b. For $a = b = 2$ and $n = 4$,from 5.12 we get,

$$|E| \leq \frac{2}{2} \cdot \sqrt[2]{\frac{2-1}{2!}} \cdot 4^{2-\frac{1}{2}} \approxeq 5.65. \tag{5.13}$$

But when we consider the actual graph with 4 vertices we can add atmost 4 edges without a $K_{2,2}$ as subgraph as shown in the figure below.

For $a = b = 2$ and $n = 5$,from 5.12 we get,

$$|E| \leq \frac{2}{2} \cdot \sqrt[2]{\frac{2-1}{2!}} \cdot 5^{2-\frac{1}{2}} \approxeq 7.9. \tag{5.14}$$

But when we consider the actual graph with 5 vertices we can add atmost 6 edges without a $K_{2,2}$ as subgraph as shown in the figure below.

Upper bound for $R(K_{a,b}, K_{a,b})$

Kővári-Sós-Turán theorem gives us a upper bound on the maximum number of edges that can be present in a $K_{a,b}$ free graph. So if the number of edges in any graph exceeds that value, we are guaranteed to get a $K_{a,b}$. We can utilize this fact to get a upper bound for $R(K_{a,b}, K_{a,b})$.

Let us assume that $R(K_{a,b}, K_{a,b}) = n$. Then the total possible number of edges is $\frac{n(n-1)}{2}$. If $\frac{n(n-1)}{2} > 2 \cdot \frac{a}{2} \cdot \sqrt[a]{\frac{b-1}{a!}} \cdot n^{2-\frac{1}{a}}$(i.e twice of that of the number of edges given by Kővári-Sós-Turán theorem), either G or G' has number of edges greater than $\frac{a}{2} \cdot \sqrt[a]{\frac{b-1}{a!}} \cdot$

$n^{2-\frac{1}{a}}$, thus always guaranteeing existence of $K_{a,b}$ in either G or G'. So we have

$$\frac{n(n-1)}{2} \geq 2 \cdot \frac{a}{2} \cdot \sqrt[a]{\frac{b-1}{a!}} \cdot n^{2-\frac{1}{a}} + 1 \tag{5.15}$$

The solution to this equation gives a upper bound for $R(K_{a,b}, K_{a,b})$. Now we can solve the equation for different values of a and b to get the upper bound on the number of vertices. We solve the equation using a piece of Matlab code attached in Appendix A.4. We take the ceiling of the solutions if they are fraction in order to get the upper bound .

Table 5.1: Upper bounds on $R(K_{a,b}, K_{a,b})$ from 5.15

b / a	1	2	3	4	5	6	7	8
1	2	4	6	8	10	12	14	16
2		11	19	27	35	43	51	59
3			75	111	147	183	219	255
4				516	687	858	1028	1199
5					3339	4172	5005	5839
6						20742	24890	29037
7							125500	146415
8								7456621

Now we can comapare these results with the bounds we have on $R(K_{a,a}, K_{a,a})$ by $R(K_{2a}, K_{2a})$. This actually gives us our best possible upper bounds.

Table 5.2: Comparision of Upper bounds on $R(K_{a,a}, K_{a,a})$ from 5.15 and 5.1

a	$R(K_{a,a}, K_{a,a})$ 5.1	$R(K_{2a}, K_{2a})$
2	11	18
3	75	165
4	516	1870

Upper bound for $R(K_{a,b}, K_{c,d})$

Just like the previous case, as Kővári-Sós-Turán theorem gives us a upper bound on the maximum number of edges that can be present in a $K_{a,b}$ free graph, if the number of edges in any graph exceeds that value, we are guaranteed to get a $K_{a,b}$. We can utilize this fact to get a upper bound for $R(K_{a,b}, K_{c,d})$ as follows.

Let us assume that $R(K_{a,b}, K_{c,d}) = n$. Then the total possible number of edges is $\frac{n(n-1)}{2}$. If $\frac{n(n-1)}{2} > \frac{a}{2} \cdot \sqrt[a]{\frac{b-1}{a!}} \cdot n^{2-\frac{1}{a}} + \frac{c}{2} \cdot \sqrt[c]{\frac{d-1}{c!}} \cdot n^{2-\frac{1}{c}}$(i.e sum of the maximum number of edges given by Kővári-Sós-Turán theorem for a $K_{a,b}$ free graph and a $K_{c,d}$ free graph), by symmetry, either G has number of edges greater than $\frac{a}{2} \cdot \sqrt[a]{\frac{b-1}{a!}} \cdot n^{2-\frac{1}{a}}$ or or G' has number of edges greater than $\frac{c}{2} \cdot \sqrt[c]{\frac{d-1}{c!}} \cdot n^{2-\frac{1}{c}}$, thus always guaranteeing existence of either $K_{a,b}$ in G or $K_{c,d}$ in G'. So we have the following equation, which if we solve for n would give us a upper bound for $R(K_{a,b}, K_{c,d})$.

$$\frac{n(n-1)}{2} \geq \frac{a}{2} \cdot \sqrt[a]{\frac{b-1}{a!}} \cdot n^{2-\frac{1}{a}} + \frac{c}{2} \cdot \sqrt[c]{\frac{d-1}{c!}} \cdot n^{2-\frac{1}{c}} + 1 \tag{5.16}$$

Chapter 6

Lower bounds for Ramsey numbers for complete tripartite 3-uniform subgraphs

Let $R'(a,b,c)$ be the minimum number n such that any n-vertex 3-uniform hypergraph $G(V,E)$, or its complement $G'(V,E)$ contains a $K_{a,b,c}$. An r-uniform hypergraph is a hypergraph where every hyperedge has exactly r vertices. (Hyperedges of a hypergraph are subsets of the vertex set. So, usual graphs are 2-uniform hypergraphs.) Here, $K_{a,b,c}$ is defined as the complete tripartite 3-uniform hypergraph with vertex set $A \cup B \cup C$, where the A, B and C have a, b and c vertices respectively, and $K_{a,b,c}$ has abc 3-uniform hyperedges $\{u,v,w\}$, $u \in A$, $v \in B$ and $w \in C$. It is easy to see that $R'(1,1,1) = 3$; with 3 vertices, there is one possible 3-uniform hyperedge which either is present or absent in G.

6.1 $R'(a,b,c)$ for small values of a,b,c

Theorem 32. $R'(1,1,2) = 4$.

Proof Consider the complete 3-uniform hypergraph with vertex set $V = \{1,2,3,4\}$ and set of exactly four hyperedges $H = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$. Since vertex 1 is present in 3 hyperedges, any (empty or non-empty) subset S of H, or its complement $H \setminus S$ must contain at least two hyperedges containing the vertex 1. Observe that any such set of two hyperedges is a $K_{1,1,2}$. $\square$

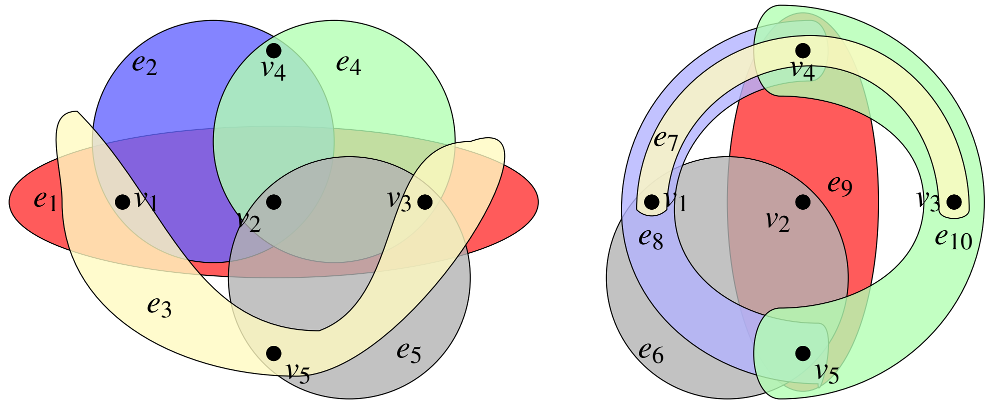

Figure 6.1: Hypergraph G(left) and its complement G'(right) each free from $K_{1,1,3}$
.

The fact that $R'(1,1,3) > 5$ can be established by counter example given in Figure 6.1.$v_1, v_2..., v_5$ represent the vertices and $e_1, ..., e_{10}$ represent the 3-uniform hyperedges. G has five hyperedges namely e_1 $(\{1,2,3\})$, e_2 $(\{1,2,4\})$, e_3 $(\{1,3,5\})$, e_4 $(\{2,3,4\})$, e_5 $(\{2,3,5\})$. G' has the rest five hyperedges namely e_6 $(\{1,2,5\})$, e_7 $(\{1,3,4\})$, e_8 $(\{1,4,5\})$, e_9 $(\{2,4,5\})$, e_{10} $(\{3,4,5\})$. $R'(1,1,4) > 6$ can be established by the following counter example that splits the $\binom{6}{3}$ edges into G and G' as follows: G ={{1,2,4}, {1,3,5}, {1,3,6}, {1,4,5}, {1,4,6}, {1,5,6}, {2,3,4}, {2,3,5}, {2,3,6}, {2,4,5}} and G' ={{1,2,3}, {1,2,5}, {1,2,6}, {1,3,4}, {2,4,6}, {2,5,6}, {3,4,5}, {3,4,6}, {3,5,6}, {4,5,6}}.

All these lower bounds are derived using a particular algorithm which we demonstrate for the particular case of $R'(1,1,3) > 5$. As there are $\binom{5}{3} = 10$ distinct 3-uniform hyperdges possible with 5 vertices, there are 2^{10} possible 3-uniform hypergraphs. We designate each of the 10 hyperedges with a distict number starting from 0 to 9. For example, hyperedge $\{1,2,3\}$ is mapped to 0 and $\{3,4,5\}$ is mapped to 9. Then we generate all possible distinct $K_{1,1,3}$ which are $\binom{5}{2} = 10$ in number. Then we generate all possible 2^{10} hypergraphs and check for existence of any of the 10 $K_{1,1,3}$. For example, edges $\{\{1,2,3\},\{1,2,4\},\{1,2,5\}\}$ denotes the presence of $K_{1,1,3}$ (0 1 2), edges $\{\{1,2,3\},\{1,3,4\},\{1,3,5\}\}$ denotes the presence of $K_{1,1,3}$ (0 4 5). For generating all possible hypergraphs, we take a 10-bit binary number, where each bit represent a particular hyperedge(0^{th} bit represent $\{1,2,3\}$ and 9^{th} bit represents $\{3,4,5\}$) and generate its all possible combinations. Now for any 10-bit binary string, we check for existence of $K_{1,1,3}$. For example, let the binary string be 000000111. It represents the hypergraph with edges $\{\{1,2,3\},\{1,2,4\},\{1,2,5\}\}$ which denotes the presence of $K_{1,1,3}$ (0 1 2). If for any hypergraph, none of the $K_{1,1,3}$ are present, then we check the existence of any of the $K_{1,1,3}$ in the complement hypergraph. The hypergraph for which both itself and its complement is free from any of the $K_{1,1,3}$ produce our counter example hypergraph.

Determining such Ramsey numbers for higher parameters by exhaustive searching using computer programs is computationally very expensive in terms or running time. We have the following upper bound for $R'(1,1,b)$.

Theorem 33. $R'(1,1,b) \leq 2b+1$.

Proof Let v_1,v_2, ..., v_{2b+1} be the $2b+1$ vertices. Then for any pair v_i,v_j, there are $2b-1$ possible 3-uniform hyperedges(each hyperedge contains one distinct vertex from the remaining vertices). So either the graph or its complement must contain b of these hyperedges containing v_i and v_j. This set of b hyperedges denote a $K_{1,1,b}$. □

We state our conjecture for $R'(1,1,b)$ as follows.

Conjecture 1. $2b \leq R'(1,1,b)$.

To settle this conjecture it is required to show that there exists some $2b-1$-vertex 3-uniform hypergraph G such that neither G nor its complement G' has a $K_{1,1,b}$. We related this problem to that of the existence of a t-design. A t-design is defined as follows. A $t-(v,k,\lambda)$ design is an incidence structure of points and blocks with properties (i) v is the number of points, (ii) each block is incident on k points, and (iii) each subset of t points is incident on λ common blocks [15].

Lemma 4. *If there is a* 2-$(2b-1,3,b-1)$ *design then* $R'(1,1,b) \geq 2b$.

Proof The existence of 2-$(2b-1,3,b-1)$ design would suggest that there exist a 3-uniform hypergraph with $2b-1$ vertices such that every pair of vertices for a hyperedge with exactly $b-1$ other vertices. This implies that the hypergraph is free of $K_{1,1,b}$. So every pair of vertices will also form a hyperedge in the complement hypergraph with exactly $b-2$ vertices. Therefore, the complement hypergraph is also free of $K_{1,1,b}$.

Table 6.1: Lower bounds for $R'(a,a,a)$ by Theorem 34 (left) and Theorem 35 (right)

a	3	4	5	6	7	8
$R'(a,a,a)$	14,19	84,138	800,1765	11773,35167	269569,1073543	9650620,50616072

Table 6.2: Lower bounds for $R'(a,b,c)$ by Theorem 34 (left) and Theorem 35 (right)

	a=2	a=3	a=3	a=3	a=4	a=4	a=5	a=6	a=6	a=6	a=6
c	5	3	4	5	4	5	5	2	3	4	5
b											
2	9,13	8,11	11,16	16,22	18,25	26,36	40,58	11,16	21,29	36,52	59,87
3	16,22	14,19	23,32	35,50	41,61	68,107	124,208		50,74	107,175	209,371
4	26,36		41,61	68,107	84,138	159,281	334,653			277,521	643,1354
5	40,58			124,208		334,653	800,1765				1740,4194

6.2 Probabilistic lower bound for $R'(a,b,c)$

Theorem 34.

$$R'(a,b,c) > \frac{\left(a^a b^b c^c \sqrt{(2\pi)^3 abc}\right)^{\left(\frac{1}{a+b+c}\right)} 2^{\left(\frac{abc-1}{a+b+c}\right)}}{e}. \tag{6.1}$$

Proof Consider the probability of existence of a particular $K_{a,b,c}$ in G or G', where G is a 3-uniform hypergraph and G' is its complement. The sum p of such probabilities over all possible distinct $K_{a,b,c}$'s is an upper bound on the probability that some $K_{a,b,c}$ exists in G or G'. Let n be the number of vertices of hypergraph G. As in the proof of Theorem 22, we observe that the number of $K_{a,b,c}$'s is no more than $\binom{n}{a} \cdot \binom{n-a}{b} \cdot \binom{n-a-b}{c}$. Each $K_{a,b,c}$ has exactly abc hyperedges. Each hyperedge can be present in G or G' with equal probability. So, the probability that all hyperedges of a particular $K_{a,b,c}$ are in G is $\left(\frac{1}{2}\right)^{abc}$. Therefore, the probability that a particular $K_{a,b,c}$ is present in either G or G' is $2 \cdot \left(\frac{1}{2}\right)^{abc} = 2^{1-abc}$. So, the probability p that some $K_{a,b,c}$ is either in G or in G', is $\binom{n}{a} \cdot \binom{n-a}{b} \cdot \binom{n-a-b}{c} \cdot 2^{1-abc}$. So choosing $n \frac{\left(a^a b^b c^c \sqrt{(2\pi)^3 abc}\right)^{\left(\frac{1}{a+b+c}\right)} 2^{\left(\frac{abc-1}{a+b+c}\right)}}{e}$, as in Inequality 6.1, and replacing $a!$ by $\sqrt{2\pi}\frac{a^{a+\frac{1}{2}}}{e^a}$, $b!$ by $\sqrt{2\pi}\frac{b^{b+\frac{1}{2}}}{e^b}$ and $c!$ by $\sqrt{2\pi}\frac{c^{c+\frac{1}{2}}}{e^c}$ as per Stirling's approximation, we get $p < 1$, guaranteeing the existence of an n-vertex graph for which some edge bicoloring would not result in any monochrmatic $K_{a,b,c}$.

$$
\begin{aligned}
p &= \binom{n}{a}\cdot\binom{n-a}{b}\cdot\binom{n-a-b}{c}\cdot 2^{1-abc}\\
&\le \frac{n^a}{a!}\cdot\frac{(n-a)^b}{b!}\cdot\frac{(n-a-b)^c}{c!}\cdot 2^{1-abc}\\
&< \frac{n^a}{a!}\cdot\frac{n^b}{b!}\cdot\frac{n^c}{c!}\cdot 2^{1-abc}\\
&\le \frac{n^{a+b+c}}{\left(\sqrt{2\pi}\frac{a^{a+\frac{1}{2}}}{e^a}\right)\cdot\left(\sqrt{2\pi}\frac{b^{b+\frac{1}{2}}}{e^b}\right)\cdot\left(\sqrt{2\pi}\frac{c^{c+\frac{1}{2}}}{e^c}\right)}\cdot 2^{1-abc}\\
&= 1.
\end{aligned}
$$

This establishes the theorem since $p < 1$ implies the existence of a hypergraph G of n vertices such that neither G nor G' has a $K_{a,b,c}$. □

See Tables 6.1 and 6.2 for some computed lower bounds based on Theorem 34 and Theorem 35.

6.3 A lower bound for $R'(a,b,c)$ using Lovász' local lemma

Theorem 35. *If* $e\cdot 2^{1-abc}\cdot\left(abc\binom{n-3}{a+b+c-3}\binom{a+b+c-3}{b-1}\binom{a+c-2}{c-1}+1\right)\le 1$, $R'(a,b,c) > n$

Proof Here we perform analysis as done earlier in Section 3.4.2. Consider a random bicoloring of the hyperedges of the complete 3-uniform hypergraph of n vertices, in which each hyperedge is independently colored red or blue with equal probability. Let S be the set of hyperedges of an arbitrary $K_{a,b,c}$, and let E_S be the event that the $K_{a,b,c}$ is coloured monochromatically. For each such S, $P(E_S) = 2^{1-abc}$. If we enumerate all possible $K_{a,b,c}$'s as S_1,S_2,...,S_m, where $m = \binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}$, each event E_{S_i} is mutually independent of all the events from the set $\{E_{S_j} : |S_i\cap S_j| = 0\}$. For each E_{S_i}, the number of events outside this set satisfies the inequality $\{E_{S_j} : |S_i\cap S_j|\ge 1\}\le abc\binom{n-3}{a+b+c-3}\binom{a+b+c-3}{b-1}\binom{a+c-2}{c-1}$, as every S_j in this set shares at least one of the abc hyperedges of S_i, and therefore S_j shares at least three vertices with S_i. We can choose the rest of the $a+b+c-3$ vertices of S_j from the remaining $n-3$ vertices, out of which we can choose $b-1$ for the second partite of S_j, and the remaining $c-1$ for the third partite of S_j, thereby yielding a $K_{a,b,c}$ which shares at least one hyperedge edge with S_i. We can apply Corollary 1 to the set of events $E_{S_1}, E_{S_2}, \ldots, E_{S_m}$, with

$$p = 2^{1-abc}, d = abc \binom{n-3}{a+b+c-3} \binom{a+b+c-3}{b-1} \binom{a+c-2}{c-1}, \tag{6.2}$$

yielding $ep(d+1) \leq 1 => Pr\left[\bigcap_{i=1}^{m} \overline{E}_{S_i}\right] > 0$. Since no event E_{S_i} occurs for some random bicoloring of the hyperedges, no monochromatic $K_{a,b,c}$ exists in that bicoloring. This establishes the theorem. □

See Tables 6.1 and 6.2 for some computed lower bounds based on Theorem 35; the values based on Theorem 35 to the right in each cell of these tables are much better than those based on Theorem 34, to the left in the respective cells.

Chapter 7

Conclusion

Ramsey theory deals with the guaranteed occurrence of specific structures in some part of a large arbitrary structure which has been partitioned into finitely many parts. Bipartite Ramsey number $R(K_{a,b}, K_{a,b})$ is just an extension from the original idea and also bounded by Ramsey number $R(a+b, a+b)$, hence it has similar characteristics as Ramsey numbers. All results in this theory typically have two primary features, their existence can be proved but they are non-constructive, and they grow exponentially. The probabilistic method is useful in establishing lower bounds for Ramsey numbers. It is worthwhile studying the application of Lovász' local lemma, possibly more effectively and accurately, so that higher lower bounds may be determined. In our work we have considered the bicoloring of K_n and the existence of a monochromatic $K_{a,b}$ in arbitrary bicolorings of the edges of K_n; some authors consider complete bipartite graphs $K_{n,n}$ instead of complete graphs like K_n and derive bounds for corresponding Ramsey numbers. There are a few open problems, some of which are highlighted below.

Open Problem 1. *Whether $R(K_{2,b}, K_{2,b})$ is equal to $4b-2$.*

Exoo et al. [7] proved that $R'(2,b) \leq 4b-2$ for all $b \geq 2$, where the equality holds if and only if a strongly regular $(4b-3, 2b-2, b-2, b-1)$-graph exists (A k-regular graph G with n vertices is called strong regular graph(n,k,p,q) if every adjacent vertices share exactly p neighbours and every non adjacent vertices share exactly q neighbours).

Open Problem 2. *Evaluating exact values for $R(K_{3,b}, K_{3,b})$.*

Exact values and upper bounds of $R(K_{3,b}, K_{3,b})$ is known for very small values of b. For example, $R(K_{3,3}, K_{3,3}) = 18$[16], $R(K_{3,4}, K_{3,4}) \leq 30$ [9], $R(K_{3,5}, K_{3,5}) \leq 38$ [9]. It is also known that $R(K_{3,b}, K_{3,b}) \leq 8b - 2$[9]. So the problem of finding the exact values and tighter bounds for this case is really worthwhile.

Open Problem 3. *Constructive tighter lower bound for $R(K_{a,b}, K_{a,b})$ and $R(K_{a,b}, K_{c,d})$.*

All the present bounds available for the general cases of $R(K_{a,b}, K_{a,b})$ and $R(K_{a,b}, K_{c,d})$ uses the probabilistic methods. Though the current bounds gives a idea about the numbers, but its is still very impossible to predict how good or bad the bound really is. So it is worthwhile to get some constructions for the lower bounds.

Open Problem 4. *Whether $R'(1,1,b) \geq 2b$.*

This is the conjecture we posed in Chapter 6. This conjecture may proven by showing the existence of a 2-$(2b-1, 3, b-1)$ design as proven by Lemma 4. Settling of the conjecture would mean that $R'(1,1,b)$ is lower bounded by $2b$ and upper bounded by $2b+1$, which would create a really tight bound for $R'(1,1,b)$.

For computing the lower bounds in Tables 1, 2 and 3, we have used computer programs. As the sizes of the complete bipartite graphs (tripartite 3-uniform hypergraphs) grow, the computation time required for computing the lower bounds becomes prohibitive. Thus, even when looking for Ramsey numbers for other small values of (a,b,c,d)'s, a new theoretical approach seems to be essential.

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Appendix A

Code for solving Inequalities 3.1,23,24, 5.15

A.1 Code for solving Inequality 3.1

File:grahamslowbnd.m

```
clc;
format longE;
e=zeros(8,8);

for a=1:8
    for b=1:8
            c=a^(0.5) * b^(0.5) * 2^(a*b-1);
            c=c*2*pi;
            d=c^(1/(a+b));
            d=d*(a+b)/exp(2);
        e(a,b)=d;
    end
end
e
```

A.2 Code for solving Inequality 23

File:lowerbound.m

```
 clc;
format longE;
e=zeros(8,8);
```

```
for a=1:8
    for b=1:8
                c=a^(a+0.5) * b^(b+0.5) * 2^(a*b-1);
                c=c*2*pi;
                d=c^(1/(a+b));
                d=d/exp(1);
        e(a,b)=d;
    end
end
e
```

A.3 Code for extracting lowerbound from Theorem 24

File:Lovaslowerbound.m

```
format longE;
e=zeros(8,8);

for a=1:8
  for b=1:8
    for n=a+b:102
      c=2^(1-a*b);
          d=nchoosek(n,a+b-2);
      g=nchoosek(a+b-2,b-1);
          f=exp(1)*c*(a*b*d*g+1);
          if(f>1)
        e(a,b)=n-1;
        break;
      end
    end
  end
end
e
```

A.4 Code for extracting upperbound from Inequality 5.15

File:Upperbound.m

```
format longE;
e=zeros(8,8);

for a=1:8
```

```
  for b=1:8
    for n=a+b:102
      c=2^(1-a*b);
          d=nchoosek(n,a+b-2);
      g=nchoosek(a+b-2,b-1);
          f=exp(1)*c*(a*b*d*g+1);
          if(f>1)
        e(a,b)=n-1;
        break;
      end
    end
  end
end
e
```

Appendix B

Code for proving $R(K_{2,3}, K_{2,3}) > 7$, $R'(1,1,3;3) > 5$ and $R'(1,1,4;3) > 6$

B.1 $R(K_{2,3}, K_{2,3}) > 7$

File:allcomb1.txt This file contains all the 210 possible $K_{2,3}$ with 7 vertices. Some of them are given below.

```
0 1 2 3 4
0 1 2 3 5
0 1 2 3 6
0 1 2 4 5
0 1 2 4 6
...
```

File:r2b7.c

```
#include<stdio.h>
#include<math.h>
#include<stdlib.h>
 int main(){
 int a,b,i,j,k,n,c,d,e,temp;
 int G[7][7];
 int G1[7][7];
 int k2bExists=0;
 int x;
 FILE *fp;
 fp=fopen("allcomb1.txt","r");
 x=pow(2,21);
 n=0;
 while(n<x){
    temp=n;
```

```
    for(i=0;i<7;i++)
     for(j=0;j<7;j++)
           G[i][j]=G1[i][j]=0;
    for(i=6;i>=0;i--)
          for(j=6;j>=0;j--){
             if(j>i){
                  G[i][j]=G[j][i]=n%2;
                  G1[i][j]=G1[j][i]=G[i][j]^1;

                  n=n/2;
             }
          }
    while(!feof(fp)){
          fscanf(fp,"%d %d %d %d %d\n",&a,&b,&c,&d,&e);
          if(G[a][c]==1&&G[a][d]==1&&G[a][e]==1&&G[b][c]==1
                      &&G[b][d]==1&&G[b][e]==1){
             k2bExists=1;
             break;
          }
         }
    if(k2bExists==0){
          rewind(fp);
          while(!feof(fp)){

             fscanf(fp,"%d %d %d %d %d\n",&a,&b,&c,&d,&e);
             if(G1[a][c]==1&&G1[a][d]==1&&G1[a][e]==1&&G1[b][c]==1
                            &&G1[b][d]==1&&G1[b][e]==1){
                  k2bExists=1;
                  break;
             }
          }
    }
    n=temp;
    printf("%d ",temp);
    n++;
    if(k2bExists==0){
    printf("\nUnsuccessful case found theorem is false for %d\n",temp);
    exit(0);
   }
   k2bExists=0;
  }
  printf("\nSuccessful theorem is true\n");
  return 0;
```

```
}
```

B.2 $R'(1,1,3;3) > 5$

File:allk113.txt

```
0 1 2
0 3 4
1 3 5
2 4 5
0 6 7
1 6 8
2 7 8
3 6 9
4 7 9
5 8 9
```

File:r113.c

```
/*
Name        :   R'(1,1,3;3)
Author      :   Tapas Kumar Mishra<tap1cse@gmail.com>
Roll        :   11CS60R32
Date        :   SUN AUG 1 16:14 IST 2012
Description :   C Program to check whether R'(1,1,3;3)>5.

*/
#include<stdio.h>
#include<stdlib.h>

int main(){

  int val=0;
  int ver[10],vercom[10];
  int i=0,temp;
  int a,b,c;
  int triexist=0;
  FILE* fp;
//------"allk113.txt" file contains all the possible K(1,1,3). Initially
//------"alltrip.txt" is created that contains all the possible
//------3-sets, that is formed lexicographically. Then from there, all
//-------the possible K(1,1,3) are extracted using careful observation.
//-------EX: the vertices are numbered 1 to 5. There are 10 possible
//------3-combinations. A one to one mapping is done between a
```

```
//———particular combination and integers 0 to 9.
//—————————0      1 2 3——————————————————————
//—————————1      1 2 4——————————————————————
//—————————2      1 2 5——————————————————————
//—————————3      1 3 4——————————————————————
//—————————4      1 3 5——————————————————————
//—————————5      1 4 5——————————————————————
//—————————6      2 3 4——————————————————————
//—————————7      2 3 5——————————————————————
//—————————8      2 4 5——————————————————————
//—————————9      3 4 5——————————————————————
//———0 1 2 in allk113.txt represents the presence of {{1,2,3},{1,2,4}
//------{1,2,5}} K(1,1,3) in Graph.
//———As there are 10 such K(1,1,3), we check for each one using
//------2^10=1024 cases. If in every case, the graph or its complement
//———contains some K(1,1,3), then we are done.
 fp=fopen("allk113.txt","r+");
 for(val=0;val<1024;val++){
      rewind(fp);
      printf("%d\t",val);
      temp=val;
      for(i=0;i<10;i++){
               ver[i]=0;vercom[i]=1;
      }
      i=0;
      while(temp>0){
               ver[i]=temp%2;
               vercom[i]=ver[i]^1;
               temp/=2;
               i++;
      }
      while(!feof(fp)){
               fscanf(fp,"%d␣%d␣%d",&a,&b,&c);
              //printf("%d %d %d\n",a,b,c);
               if((ver[a]==1 && ver[b]==1 && ver[c]==1)||
                  (vercom[a]==1 &&  vercom[b]==1 && vercom[c]==1))
                     {
                          triexist=1;
                          break;
                     }
      }

      if(triexist==0){
```

```
                printf("failed-for-value:%d\n",val);
                exit(0);
        }
        triexist=0;
}
fclose(fp);
printf("\nsuccess\n");
return 0;
}
```

B.3 $R'(1,1,4;3) > 6$

File:allk114.txt

```
0 1 2 3
0 4 5 6
1 4 7 8
2 5 7 9
3 6 8 9
0 10 11 12
1 10 13 14
2 11 13 15
3 12 14 15
4 10 16 17
5 11 16 18
6 12 17 18
7 13 16 19
8 14 17 19
9 15 18 19
```

File:r114.c

```
/*
Name        :  R'(1,1,4)
Author      :  Tapas Kumar Mishra<tap1cse@gmail.com>
Roll        :  11CS60R32
Date        :  SUN  AUG 1  20:00 IST 2012
Description :  C Program to check whether R'(1,1,4;3)>6.

*/
#include<stdio.h>
#include<stdlib.h>
#include<math.h>
int main(){
```

```
int val=0;
int ver[20],vercom[20];
int i=0,temp;
int a,b,c,d,x;
int triexist=0;
//-----"alltrip_ver1.txt" file contains all the possible K(1,1,4).
//------Initially "alltrip.txt" is created that contains all the
//------possible 3-sets, that is formed lexicographically. Then
//------from there, all the possible K(1,1,4) are extracted
//------using careful observation. EX : the vertices are numbered
//-----1 to 6. There are 20 possible 3-combinations. A one to
//------one mapping is done between a particular
//------combination and integers 0 to 9.
//---------------------0 1 2 3------------------------------------
//---------------------1 1 2 4------------------------------------
//---------------------2 1 2 5------------------------------------
//---------------------3 1 2 6------------------------------------
//---------------------4 1 3 4------------------------------------
//---------------------5 1 3 5------------------------------------
//---------------------6 1 3 6------------------------------------
//---------------------7 1 4 5------------------------------------
//---------------------8 1 4 6------------------------------------
//---------------------9 1 5 6------------------------------------
//---------------------10 2 3 4------------------------------------
//---------------------11 2 3 5------------------------------------
//---------------------12 2 3 6------------------------------------
//---------------------13 2 4 5------------------------------------
//---------------------14 2 4 6------------------------------------
//---------------------15 2 5 6------------------------------------
//---------------------16 3 4 5------------------------------------
//---------------------17 3 4 6------------------------------------
//---------------------18 3 5 6------------------------------------
//---------------------19 4 5 6------------------------------------
//-----0 1 2 3 in alltrip_ver1.txt represents the presence of
// -----{{1,2,3},{1,2,4},{1,2,5},{1,2,6}} K(1,1,4) in Graph.
//------As there are 15 such K(1,1,4), we check for each one
//------using 2^15 cases. If in every case, the graph or its
//------ complement contains some K(1,1,4), then we are done.
FILE *fp=fopen("alltrip_ver.txt","r");
x=pow(2,20);
for(val=0;val<x;val++)    {
    printf("%d\t",val);
```

```
        rewind(fp);
        temp=val;
        printf("%d\t",val);
        for(i=0;i<20;i++){
             ver[i]=0;vercom[i]=1;
        }
        i=0;
        while(temp>0){

             ver[i]=temp%2;
             vercom[i]=ver[i]^1;
             temp/=2;
             i++;
        }
          while(!feof(fp)){
             fscanf(fp,"%d %d %d %d\n",&a,&b,&c,&d);
             if((ver[a]==1&&ver[b]==1&&ver[c]==1&&ver[d]==1)||
                 (vercom[a]==1&&vercom[b]==1&&vercom[c]==1&&vercom[d]==1))
                            {
                               triexist=1;break;
              }
          }
          if(triexist==0){
               printf("failed-for-value:%d\n",val);
               exit(0);
          }
          triexist=0;
 }
 fclose(fp);
 printf("\nsuccess\n");
 return 0;
}
```