Primal-dual schema

Complementary slackness conditions

P: $\min \sum_{j=1}^{n} c_j x_j$

s.t. $\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad (i=1,\dots,m)$ $\quad y_i$

$x_j \geq 0 \quad (j=1,\dots,n)$

D: $\max \sum_{i=1}^{m} b_i y_i$

s.t. $\sum_{i=1}^{m} a_{ij} \cdot y_i \leq c_j \quad (j=1,\dots,n)$

$y_i \geq 0 \quad (i=1,\dots,m)$

Let $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_m)$ be a primal-feasible and a dual-feasible point respectively.

$$\sum_{i=1}^{m} b_i y_i \leq \sum_{i=1}^{m} y_i \left(\sum_{j=1}^{n} a_{ij} x_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} y_i a_{ij} x_j$$

$$= \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} a_{ij} y_i\right) \leq \sum c_j x_j$$

Let $x^* = (x_1^*,\dots,x_n^*)$ & $y^* = (y_1^*,\dots,y_m^*)$ be a primal optimum point & a dual optimum point respectively.

strong duality $\Rightarrow \sum_{i=1}^{m} b_i y_i^* = \sum_{j=1}^{n} c_j x_j^*$

Thus, both the inequalities in the above chain are tight when $y = y^*$ & $x = x^*$.

$$\therefore \sum_{i=1}^{m} b_i y_i^* = \sum_{i=1}^{m} y_i \sum_{j=1}^{n} a_{ij} x_j^* \quad \text{— ①, and}$$

$$\sum_{j=1}^{n} x_j^* \sum_{i=1}^{m} a_{ij} y_i^* = \sum_{j=1}^{n} c_j x_j^* \quad \text{— ②}$$

From ① we have that,

$$\underbrace{\sum_{i=1}^{m} \underbrace{y_i^*}_{\geq 0} \cdot \underbrace{\left(b_i - \sum_{j=1}^{n} a_{ij} x_j^*\right)}_{\leq 0}}_{\leq 0} = 0$$

$$\Rightarrow \forall i = 1, \ldots, m$$

Either $y_i^* = 0$ or $\sum_{j=1}^{n} a_{ij} x_j^* = b_i$ } Dual complementary slackness conditions

Either a dual variable is 0 or the corresponding primal constraint is tight.

Similarly from ② we have.

$\forall j = 1, \ldots, n$

Either $x_j^* = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Primal complementary slackness conditions.

Relaxed complementary slackness conditions.

<u>Claim 1:</u> Let $\alpha, \beta \geq 1$. Assume that x is primal feasible, y is dual feasible, and $\forall j = 1, \ldots, n$

$$x_j \neq 0 \Rightarrow \sum_{i=1}^{m} a_{ij} y_i \geq \frac{c_j}{\alpha}$$

$$\Rightarrow c_j \leq \alpha \sum_{i=1}^{m} a_{ij} y_i$$

and,

$\forall i = 1, \ldots, m$

$$y_i \neq 0 \Rightarrow \sum_{j=1}^{n} a_{ij} x_j \leq b_i \cdot \beta$$

then, $\displaystyle\sum_{j=1}^{n} c_j x_j \leq \alpha\beta \cdot \sum_{i=1}^{m} b_i x_i$

<u>Proof:</u>

$$\sum_{\substack{j=1 \\ x_j \neq 0}}^{n} c_j x_j \leq \alpha \sum_{\substack{j=1 \\ x_j \neq 0}}^{n} \sum_{\substack{i=1 \\ y_i \neq 0}}^{m} a_{ij} y_i x_j$$

$$= \alpha \sum_{i=1}^{m} y_i \cdot \sum_{j=1}^{n} a_{ij} x_j$$

$$\leq \alpha \cdot \sum_{i=1}^{m} y_i \cdot \beta \cdot b_i$$

$$= \alpha\beta \cdot \sum_{i=1}^{m} b_i y_i .$$

$\blacksquare$

$$\sum_{j=1}^{n} c_j x_j \leq \alpha\beta . \sum_{i=1}^{m} b_i y_i \leq \alpha\beta . \sum_{i=1}^{m} b_i y_i^*$$

$$= \alpha\beta . \sum c_j x_j^* .$$

[y^* : A dual optimal pt, x^* : A primal optimal point]

Primal dual schema

- Assume that Π is a minimization problem that can be formulated as an integer program.
- Let P be a relaxation of that ILP, & let D be the dual of P.
- The algorithm iteratively constructs an integral feasible point of P (let's call it x) and a feasible point of D (let's call it y) such the pair (x, y) satisfy the relaxed complementary slackness conditions with parameters α, β.
- Claim 1 implies that x is an $\alpha\beta$-approximate solution.

The set-cover problem

Input: $U = \{1, \dots, n\}$, $\mathcal{S} = \{S_1, \dots, S_m\}$

P: $c : \mathcal{S} \to \mathbb{R}^{\geq 0}$.

$$\min \sum_{i=1}^{m} c(S_i) \cdot x_i \qquad \Bigg| \qquad \max \sum_{j=1}^{n} y_j$$

y_j s.t. $\displaystyle\sum_{i: j \in S_i} x_i \geq 1 \quad \forall j = 1, \dots, n$ | s.t. $\displaystyle\sum_{j \in S_i} y_j \leq c(S_i) \quad i = 1, \dots, m$

$x_i \geq 0 \quad \forall i = 1, \dots, m.$ | $y_j \geq 0, \quad \forall j = 1, \dots, n$

A primal-dual f-approx. algo. for set cover

(f: max. frequency of an element)

$x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$.

- $x \leftarrow 0^m$, $y \leftarrow 0^n$
- while there exists an uncovered element j
    - raise y_j until some dual constraint becomes tight.
    - Pick all tight sets (i.e., set the corresponding entries in x to 1).
- Return x.

<u>Observe</u>: ① $x_i \in \{0,1\}$ $\forall$ $i=1,\dots,m$.

② The x returned is primal feasible $\Big\}$ The algo returns a valid set-cover

③ y is always dual feasible.

$$(y_j \neq 0) \Rightarrow \left(\sum_{i: j \in S_i} x_i \leq f \cdot 1 \right)$$

$\underline{\underline{\beta = f}}$.

$(x_i \neq 0) \Rightarrow$ The algorithm has picked S_i

$\Rightarrow$ S_i is tight

$\Rightarrow$ $\sum_{j \in S_i} y_j = c(S_i)$ in the iteration in which S_i is picked, and hence also later

$\underline{\underline{\alpha = 1}}$ $\boxed{\alpha\beta = f}$

$\Rightarrow$ By Claim 1, the algorithm is an f-approximation algorithm.