Randomized lower bound for online paging.

Theorem: Let A be a randomized online paging algorithm. Then competitive ratio of A is at least H_k.

Proof: Consider the following randomized request sequence $\sigma = \sigma_1 \sigma_2 \sigma_3 \cdots \sigma_n$ where for $i = 1, \cdots, n$ σ_i is an uniform random independent sample from $\{1, 2, \cdots, k+1\}$.

For a fixed i,

$$\Pr[A \text{ incurs a miss on } \sigma_i] = \frac{1}{k+1}.$$

$$\mathbb{E}_{A,\sigma}[\text{Cost}(A,\sigma)] = \frac{n}{k+1}.$$

[Define $X_i = \begin{cases} 1 & \text{if } A \text{ incurs a miss on } \sigma_i \\ 0 & \text{otherwise} \end{cases}$

and apply linearity of expectation on $\sum_{i=1}^{n} X_i$]

Consider the following answering strategy (offline):

In the event of a miss, expel the unique page that will not be requested in the rest of the current phase.

$OPT \leq$ Number of misses by the above strategy

$\leq$ Number of phases.

$\mathbb{E}[OPT] \leq \mathbb{E}[\text{Number of phases}]$.

&

Number of requests is n.

What is the expected length of a phase.

In each step, a page from $\{1, \cdots, k+1\}$ is sampled uniformly at random. In expectation, how many pages are sampled before k distinct pages are sampled.

Let T_i be the number of samples drawn between the first occurence (exclude) of the $(i-1)^{st}$ distinct page and the first occurence (include) of the i-th distinct page $(i = 0, \cdots, k+1)$

$$\text{Length of the phase} = \sum_{i=1}^{k+1} T_i - 1$$

$$\Rightarrow \mathbb{E}[\text{Length of the phase}] = \sum_{i=1}^{k+1} \mathbb{E}[T_i] - 1.$$

$$\mathbb{E}[T_1] = 1, \quad \mathbb{E}[T_2] = \frac{k+1}{k}.$$

$$\mathbb{E}[T_i] = ?$$

$$\Pr[\text{A sample is a new distinct page}] = \frac{k+1-(i-1)}{k+1} = \frac{k-i+2}{k+1}.$$

$$\therefore \mathbb{E}[T_i] = \frac{k+1}{k-i+2}$$

(Consider a coin whose probability of head is p. Expected no. of independent tosses until the first head is $\frac{1}{p}$).

$$\therefore \overbrace{\mathbb{E}[\text{Length of a phase}]}^{\mu} = \sum_{i=1}^{k+1} \mathbb{E}[T_i] - 1$$

$$= (k+1) \cdot \sum_{i=1}^{k+1} \frac{1}{k-i+2} - 1$$

$$= (k+1) \cdot H_{k+1} - 1$$

$$\mathbb{E}[\text{Number of phases}] \underset{\text{want to say}}{\approx} \frac{n}{\mathbb{E}[\text{Length of a phase}]}.$$

Let L_i be the length of the i-th phase. $(i = 1, \ldots)$

Let $\ell = \max\{i : \text{The } i\text{-th phase starts on or before the } n\text{-th request}\}$

Consider $\sum_{i=1}^{\ell} L_i$.

ℓ is the unique int. for which ① is true.

$$\sum_{i=1}^{\ell-1} L_i < n \leq \sum_{i=1}^{\ell} L_i \quad \text{— ①}$$

$$\mathbb{E}\left[\sum_{i=1}^{\ell} L_i - n\right] \leq \mu$$

$$\Rightarrow \mathbb{E}\left[\sum_{i=1}^{\ell} L_i\right] \leq n + \mu. \quad \text{— ②}$$

Define $I_i = \begin{cases} 1 & \text{if } i \leq \ell \\ 0 & \text{otherwise} \end{cases}$

Then, $\sum_{i=1}^{\ell} L_i = \sum_{i=1}^{n} I_i \cdot L_i$

$$\mathbb{E}\left[\sum_{i=1}^{\ell} L_i\right] = \sum_{i=1}^{n} \mathbb{E}\left[I_i \cdot L_i\right]$$

$$\mathbb{E}[I_i \cdot L_i] = \Pr[I_i = 1] \cdot \mathbb{E}[L_i \mid I_i = 1]$$
$$+ \Pr[I_i = 0] \cdot \underbrace{\mathbb{E}[0 \mid I_i = 0]}_{0}$$

$$= \Pr[I_i = 1] \cdot \mathbb{E}[L_i \mid I_i = 1]$$
$$= \Pr[I_i = 1] \cdot \mathbb{E}[L_i]$$
$$= \Pr[I_i = 1] \cdot \mu.$$

$$\mathbb{E}\left[\sum_{i=1}^{\ell} L_i\right] = \mu \cdot \sum_{i=1}^{n} \Pr[I_i = 1]$$
$$= \mu \cdot \sum_{i=1}^{n} \mathbb{E}[I_i]$$
$$= \mu \cdot \mathbb{E}\left(\sum_{i=1}^{n} \mathbb{E}[I_i]\right)$$
$$= \mu \cdot \mathbb{E}[\ell] \quad \text{——— } (3)$$

From (2) & (3) we have

$$n + \mu \geq \mu \cdot \mathbb{E}[\ell]$$
$$\Rightarrow \mathbb{E}[\ell] \leq \frac{n}{\mu} + 1.$$

$$= \frac{n}{(k+1)H_{k+1} - 1} + 1$$

$$\frac{\underset{A,\delta}{\mathbb{E}}[\text{Cost}(A,\delta)]}{\underset{\delta}{\mathbb{E}}[OPT]} \geq \frac{n/(k+1)}{\left(\frac{n}{(k+1)H_{k+1} - 1} + 1\right)} := \alpha$$

$$(k+1)\cdot H_{k+1} - 1$$
$$= (k+1)\cdot H_k$$

$$\alpha = \frac{n/(k+1)}{\frac{n}{(k+1)\cdot H_k} + 1}$$

$$= \frac{n \cdot H_k}{n + (k+1)\cdot H_k} = \frac{H_k}{1 + \frac{(k+1)\cdot H_k}{n}}$$

$$\mathbb{E}[\text{Cost}(A,\delta)] \geq \alpha \cdot \mathbb{E}[OPT] \quad \text{— (1)}$$

independent of n.

Let A be α'-competitive.

$\Rightarrow \exists\, b$ s.t. $\forall\, \delta$.

$$\text{Cost}(A,\delta) \leq \alpha' \cdot OPT + b$$

$$\Rightarrow \mathbb{E}[\text{Cost}(A,\delta)] \leq \alpha' \cdot \mathbb{E}[OPT] + b \quad \text{— (2)}$$

From 1 & 2,

$$\alpha \cdot \mathbb{E}[OPT] \leq \alpha' \cdot \mathbb{E}[OPT] + b$$

$$\Rightarrow (\alpha - \alpha') \cdot \mathbb{E}[OPT] \leq b.$$

$$\Rightarrow \left(\frac{H_k}{1 + \frac{(k+1)}{n} \cdot H_k} - \alpha' \right) \cdot \mathbb{E}[OPT] \leq b. \quad \text{— } 3.$$

If $\alpha' < H_k$, then $\left(\frac{H_k}{1 + \frac{(k+1)}{n} \cdot H_k} - \alpha' \right) > 0$

if n is large enough. Since, b is independent of n and $\mathbb{E}[OPT]$ is a growing fn of n, the LHS of 3 can be made greater than b by choosing a large enough n. This contradicts 3.

$$\Longrightarrow \alpha' \geq H_k.$$

$\blacksquare$