Vertex-cover

Input: An undirected graph of n vertices and m edges.

problem: To find a smallest vertex cover.

Lower bounding scheme:

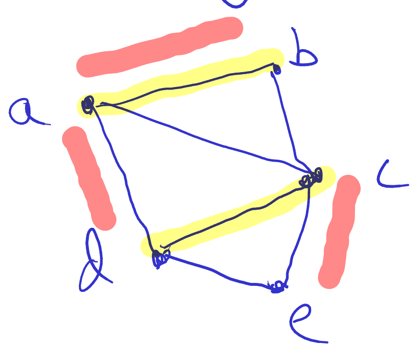
1. Find a "simple" quantity m that lower bounds OPT. $OPT \geq m$

2. Design an algorithm A that outputs a solution with cost $\leq \alpha \cdot m$

$$\forall I, \quad c(A(I)) \leq \alpha \cdot m \leq \alpha \cdot OPT$$

$$\frac{c(A(I))}{OPT} \underset{(1)}{\leq} \frac{c(A(I))}{m} \underset{(2)}{\leq} \alpha .$$

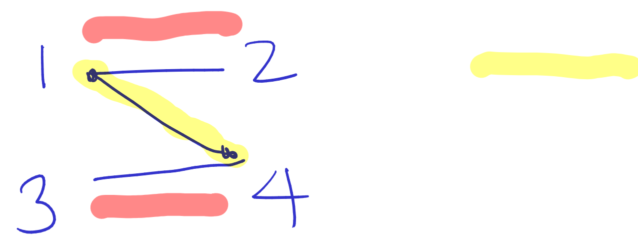
For an undirected graph $G = (V, E)$, a matching M is a subset of E consisting of vertex-disjoint edges.



Not a matching.

Is a matching.

A <u>maximal</u> matching of a graph is a matching to which no more edge can be added without violating the matching property. A <u>maximum</u> matching is a matching with maximum number of edges.



maximal but not maximum.

Let M be a maximal matching of G. Then,

$$OPT \geq |M|$$

Size of any vertex cover $\geq$ Size of any maximal matching

Proof. One endpoint of each edge of the matching must belong to the vertex-cover. $\blacksquare$

Maximal matching can be easily found by a "greedy" algorithm.

<u>Algorithm:</u>

1. Find a maximal matching M.
2. A := set of end-points of all edges in M. Output A.

<u>Proof of correctness</u>: Let $(u,v) \in E$. Towards a contradiction assume that neither u nor v is an end-point of some edge in M. Then $M \cup \{(u,v)\}$ is a matching. This contradicts the maximality of M. ☐

<u>Derivation of approximation ratio</u>:

$$|A| = 2|M| \leq 2\,\mathrm{OPT}.$$

1. Can the guarantee of this algorithm be improved by a tighter analysis?
   - No. (G is a matching of n edges).

2. Can we design an algorithm with a better factor using the size of a ~~maximal matching~~ maximum matching as the lower bounding scheme?

   Does there exist an algo A s.t.

$$\frac{c(A(I))}{|\text{maximum matching}|} \leq 2 - \varepsilon \quad [\varepsilon = \Omega(1)] \;?$$

No. We will exhibit a family of graphs for which

$$\frac{OPT}{|\text{maximum matching}|} = 2. \Rightarrow \frac{c(A(I))}{|\text{maximum matching}|} \geq 2.$$

Family: Complete graphs on n (vertices).

Size of a maximum matching $= \frac{n-1}{2}$.

$OPT = n-1$.

3. Does there exist an algorithm with approximation ratio $\leq 2 - \Omega(1)$?

— Open. Assuming "Unique Games Conjecture", No.

Kőnig's theorem:

Maximum matching = Minimum vertex cover.

Set-Cover: Input: $U = \{1, \ldots, n\}$, $S_1, \ldots, S_\ell \subseteq U$, $S_1 \cup \cdots \cup S_\ell = U$, $c: \{S_1, \ldots, S_\ell\} \longrightarrow \mathbb{R}^{\geq 0}$. Task: Find a subset $\mathcal{S}$ of

$\{S_1, \ldots, S_\ell\}$ such that: 1 $\bigcup_{S \in \mathcal{S}} S = \mathcal{U}$, 2 $\sum_{S \in \mathcal{S}} c(S)$ is minimum subject to 1.