Linear Programming

minimize $7x_1 + x_2 + 5x_3$

s.t. $x_1 - x_2 + 3x_3 \geq 10$.

L: $5x_1 + 2x_2 - x_3 \geq 6$.

$x_1, x_2, x_3 \geq 0$.

$7 \times 2 + 1 + 5 \times 3$
$= 30$.

Given (L, b) is $val(L) \leq b$?

For $b = 30$, "Yes". $\underbrace{x = (2, 1, 3)}_{\text{certificate}}$

$$\boxed{\begin{array}{l} 7x_1 + x_2 + 5x_3 \\ \geq x_1 - x_2 + 3x_3 \\ \geq 10. \text{ For every} \\ \text{feasible } x_1, x_2, x_3 \end{array}}$$

Given (L, b), is $val(L) \geq b$?

For $b = 10$, "Yes".

$$7x_1 + x_2 + 5x_3 \geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3)$$

$$\geq 10 + 6 = 16.$$

"Yes", for $\boxed{b = 16}$

Let $a, b \geq 0$ s.t.

$$7x_1 + x_2 + 5x_3 \geq a(x_1 - x_2 + 3x_3) + b(5x_1 + 2x_2 - x_3)$$

Then,

$Val(L) \geq \underbrace{10a + 6b}$.

In the first case: $a = 1, b = 0$
$Val(L) \geq 10$.

In the second case: $a = b = 1$.
$Val(L) \geq 16$.

$$\begin{aligned} \max \quad & 10a + 6b \\ \text{s.t.} \quad & a + 5b \leq 7 \\ & -a + 2b \leq 1 \\ & 3a - 5 \leq 5 \\ & a, b \geq 0 \end{aligned}$$

: L'

$\boxed{Val(L) \geq Val(L')}$. (1)

Want: A short certificate that $Val(L) \geq b$ for all such b. The above method "works" for $b \leq val(L')$.

If (1) is strict, then for a $b \in (Val(L'), Val(L))$

Not true

the above method "does not work".

L' is the **dual linear program** of L.

Inequality (1) is the **weak duality theorem**:

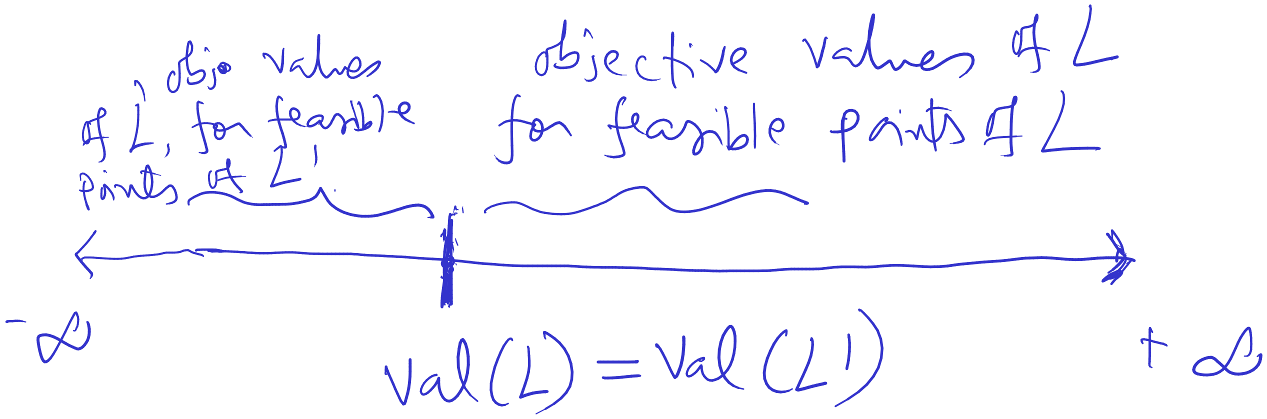
L:
$$\min \sum_{i=1}^{n} c_i x_i$$
$$\text{s.t. } a_{11}x_1 + \cdots + a_{1n}x_n \geq b_1$$
$$\vdots$$
$$a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i$$
$$\vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n \geq b_m$$

$L' \xleftarrow[\text{(exercise)}]{\text{Write the dual}} L$

Ex 1: Write the dual L'

Ex 2: Prove weak duality, i.e.,
$val(L) \geq val(L')$.

LP duality theorem: $\overbrace{val(L)} = \overbrace{val(L')}$.



L: Primal LP

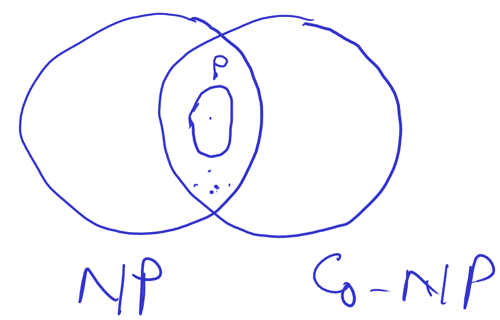
L': Dual LP

$$L: \quad \begin{array}{l} \min \ x \\ \text{s.t.} \\ -x \geq 0 \end{array} \qquad \mathrm{val}(L) = -\infty \qquad \boxed{\begin{array}{r} x_1 \geq 1 \\ x_2 \geq 1 \\ -x_1 - x_2 \geq 1 \end{array}}$$

<u>LP-duality theorem</u>: The primal has a finite optimum if and only if the dual has a finite optimum. In that case $\mathrm{val}(L) = \mathrm{val}(L')$.

LP-duality theorem $\Rightarrow$ Decision version(s) of LP $\in$ Co-NP

(Decision version) LP $\in$ NP $\cap$ Co-NP.



Linear Programs can be solved in polynomial time.

<u>Connection to vertex cover.</u> $G = (V, E)$, $V = \{1, \ldots, n\}$, m edges.

Vertex cover: A subset $A \subseteq V$ that "covers" all edges.

For each $v \in V$,
define a variable $x_v \in \{0,1\}$. n variables

$$\text{minimize } \sum_{v \in V} x_v$$

subject to:

$\forall (u,v) \in E$ $\quad x_u + x_v \geq 1$ m constraints

$\underbrace{x_v \in \{0,1\}}$: n constraints

L: $\boxed{0 \leq x_v \leq 1}$ $\quad -x_v \geq -1 \quad \forall v \in V$; $\quad x_v \geq 0 \quad \forall v \in V$

$\boxed{x_v \in \mathbb{Z}}$ Integers

Integrality Constraints

Integer Linear Program (ILP): $val(L) =$ size of a minimum VC.

An optimum feasible point $(x_1^*, \dots, x_n^*)$ defines a minimum vertex cover.

(Decision versions of) ILP $\in$ NP.

ILP is NP-hard.

Idea ① ~~Relax~~ drop the integrality constraint. Let L_1 be the resultant linear program.

$$L': \quad \min \sum_{v \in V} x_v$$
$$\text{s.t.} \quad x_u + x_v \geq 1 \quad \forall (u,v) \in E.$$
$$-x_v \geq -1 \quad \forall v \in V$$
$$x_v \geq 0.$$

$$\mathrm{val}(L') \leq \mathrm{val}(L).$$

(2) Use a LP solver to solve L' and obtain an optimal feasible point $(x_1^*, \dots, x_n^*)$.

$$\mathrm{Val}(L') = \sum_{i=1}^{n} x_i^* \leq \mathrm{Val}(L) = OPT.$$

(3) <u>Rounding</u>: $\underbrace{(x_1^*, \dots, x_n^*)}_{\text{Fractional optimum}} \xrightarrow{\leq \alpha \text{ blow-up}} \underbrace{(y_1^*, \dots, y_n^*)}_{\text{A } L\text{-feasible point}} = \bar{y}^*$

$\bar{y}^*$ defines a set $A = \{v \in V : y_v^* = 1\}$. A is a vertex cover.

$$|A| = \sum_{i=1}^{n} y_i^* \leq \alpha \cdot \sum_{i=1}^{n} x_i^* = \alpha \cdot \mathrm{val}(L') \leq \alpha \cdot OPT.$$

Rounding:

$\forall v \in V, \quad y_v^* := \begin{cases} 1 & \text{if } x_v^* \geq \frac{1}{2} \\ 0 & \text{otherwise}. \end{cases}$

Check: 1. $\overline{y^*}$ is L-feasible

2. $\sum y_v^*$ with $\sum x_v^*$.

2. $y_v^* \leq 2 \cdot x_v^* \Rightarrow \sum_{v \in V} y_v^* \leq 2 \cdot \sum x_v^*$. ✓

1. Each $y_v^* \geq 0$ ✓. Each $-y_v^* \geq -1$ ✓.

$(u,v) \in E$.

$x_u^* + x_v^* \geq 1$.

$\Rightarrow$ Either x_u^* or x_v^* is at least $\frac{1}{2}$.

$\Rightarrow$ Either y_u^* or y_u^* is 1.

$\Rightarrow y_u^* + y_v^* \geq 1$. ✓

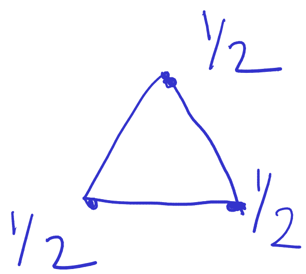
LP-based approximation algorithms

→ Rounding.

→ Primal-dual schema.

Exercise 3: Write the dual of the relaxation of the vertex cover LP. See, if you can interpret it combinatorially.

Exercise 4: Write the dual of dual of an LP, and see what you get.



- Size of an optimal vertex cover $= 2$.
- Value of the relaxation $\leq \frac{3}{2}$.

$=$

Set-cover: Input: $\mathcal{U} = \{1, \dots, n\}$, $S_1, \dots, S_m \subseteq \mathcal{U}$.

$S_1 \cup \dots \cup S_m = \mathcal{U}$. $c : \{S_1, \dots, S_m\} \to \mathbb{R}^{\geq 0}$.

Task: Find a collection of sets from $S_1, \dots, S_m$ whose union is $\mathcal{U}$ of minimum total cost

(1) For $i \in \{1, \dots, m\}$, $x_i \in \{0, 1\}$.

(2) minimize $\sum_{i=1}^{m} c(S_i)\cdot x_i$

L: $\sum_{i: j \in S_i} x_i \geq 1 \quad \forall j \in U$

$x_i = 1$ for some i s.t. $j \in S_i$

$\underbrace{x_i \in \{0,1\}}$

$x_i \geq 0$

$-x_i \geq 1$

$x_i \in \mathbb{Z}$

(2) Relax the above ILP; remove the constraints $x_i \in \mathbb{Z}$.
Let the resultant LP be L'.

(3) Solve L' and obtain an optimum feasible point $(x_1^*, \ldots, x_m^*)$.

(4) Define $y_i^* = \begin{cases} 1 & \text{if } x_i^* \geq \frac{1}{f} \\ 0 & \text{otherwise} \end{cases}$

observation: $\forall i,\ y_i^* \leq f \cdot x_i^*$

Output $\{S_i : y_i^* = 1\}$.

<u>Analysis</u>: Cost of the returned collection

$$= \sum_{i=1}^{m} y_i^* \cdot c(S_i) \leq f \cdot \sum_{i=1}^{m} x_i^* \cdot c(S_i)$$

$$= f \cdot OPT(L') \leq f \cdot OPT.$$

Let $j \in U$. $\sum_{i : j \in S_i} x_i^* \geq 1$ (From the feasibility of $(x_1^*, \ldots, x_m^*)$)

The number of terms in the above summation is at most f. This implies that there exists an i s.t. $j \in S_i$ and $x_i^* \geq \frac{1}{f}$.

$$\Rightarrow y_i^* = 1. \Rightarrow \sum_{i : j \in S_i} y_i^* \geq 1.$$

$\Rightarrow (y_1^*, \ldots, y_2^*)$ is feasible for L. $\blacksquare$