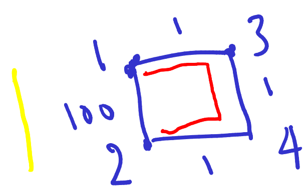
Steiner Tree and Travelling Salesman problem

Steiner Tree problem:

Input: An undirected weighted graph G=(V, E). Each edge e has a non-negative cost c(e). A bipartition of V into a set of "required vertices" R, and a set of "Steiner vertices" S. R ∩ S is empty, and R U S = V. Find a minimum cost tree that contains all the vertices in R, and a subset (possibly empty) of vertices from S.



$R = \{1, 2\}$

Travelling Salesman Problem:

Input: is a completed undirected graph G. Each edge has a on-negative cost c(e). Find a cycle that visits each vertex exactly once (i.e. a Hamiltonian cycle) of minimum total weight.

Metric version: Complete graph. Edge costs satisfy traingle inequality. For every u, v, w in V,) $c(u,v) + c(v,w) \geq c(u,w)$.

Steiner tree $\underbrace{\text{reduces to}}_{\text{approximation factor preserving reduction.}}$ Metric Steiner tree problem

Given: Complete undirected graph G, and cost function satisfies Δ-inequality.

Claim 1: If there exists a polytime α-approximation algorithm for the Metric Steiner Tree problem, then that can be used to construct a poly time α-approximation algo for the general Steiner tree problem.

Claim 2: Let α be a polytime computable (*) function, and P ≠ NP. Then there does not exist a α(n)-factor polytime approximation algo for TSP.

[(*) There is an algo that, given n, computes $\alpha(n)$ in poly(n) time.]

For metric TSP, there exists a $\frac{3}{2}$-approximation algorithm.

Proof of claim 1: Let A be a polytime α-approximation algorithm for the metric Steiner tree problem. Let $(G=(V,E), c)$ be an instance for the (general) Steiner tree problem.

Step 1: Let G' be a complete graph with vertex set V. Define $c'(u,v) :=$ ~~Length~~ sum of cost of edges of a shortest path from u to v in G (w.r.t. c). (G', c') is called the metric closure of G.

v w in G

u

There is a path from u to w of total cost $c'(u,v) + c'(v,w)$.

$$c'(u,w) \le c'(u,v) + c'(v,w).$$

<u>claim</u>: $OPT_{G'} \leq OPT_G$.

Consider an optimal steiner tree T of G. T is also a Steiner tree of G'. For $(u,v) \in E$, $c'(u,v) \leq c(u,v)$. — (1)

$\Rightarrow OPT_{G'} \leq c'(T) \underset{(1)}{\leq} c(T) = OPT_G$.

<u>Step 2</u>: Run A on (G', c', R, S). Let T be the output.

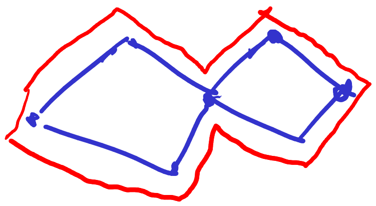
Replace each edge (u,v) in T with a shortest path from u to v in G. Call the resultant graph T'.

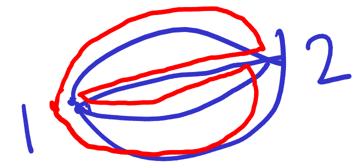
[If (G,c) has a steiner tree, then T does not have any edge with infinite cost. So, u & v are connected in T]

T' contains all vertices in R. By removing edges if necessary, turn T' into a tree T''.

$$c(T'') \leq c(T') = c'(T) \leq \alpha \cdot OPT_{G'} \leq \alpha \cdot OPT_G.$$

An Euler circuit of a (multi)graph is a closed path that traverses each edge exactly once.



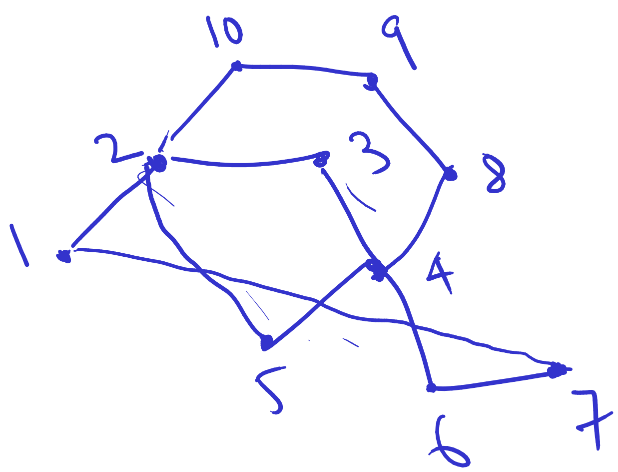


$\leftarrow \deg(1) = 4 = \deg(2)$.

a graph has an Euler circuit $\Longleftrightarrow$ Degree of each vertex is even.

exercise

(given a multigraph in which the degree of each vertex is even, an Euler circuit can be constructed in polynomial time).



C_1: 1 2 3 4 5 2 10 9
8 4 3 2 5 4 6 7 1

$\Downarrow$

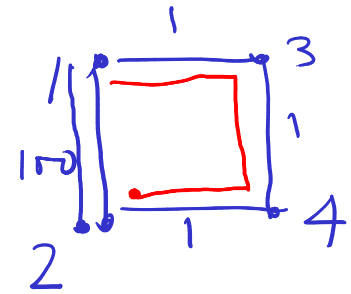
C_2: 1 2 3 4 5 × 10 9 8 ×
× × × × 6 7 1.

<u>claim</u> : $c(C_2) \le c(C_1)$.

Metric Steiner Tree

Input: Complete graph $G = (V, E)$, $c: E \to \mathbb{R}^{\geq 0}$ satisfies $\triangle$-inequality, $R \subseteq V$ ("required set"). $S := V \setminus R$ (steiner vertices).

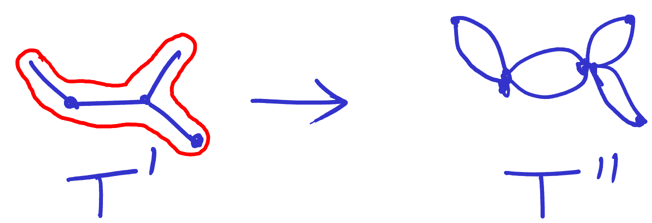
Algorithm: - Return an MST T of the subgraph of G induced by R.



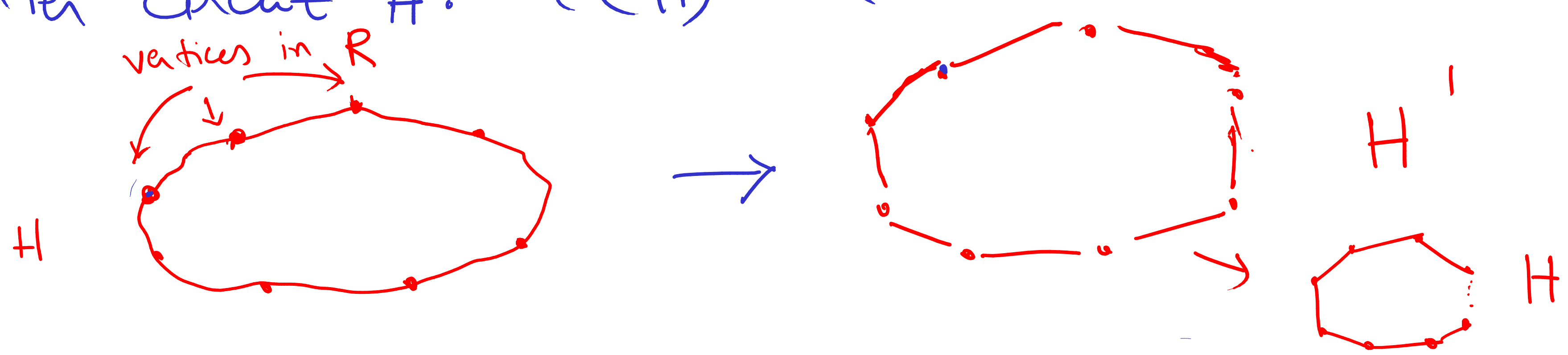
$R = \{1, 2\}$.

Claim: $c(T) \leq 2 \cdot OPT$

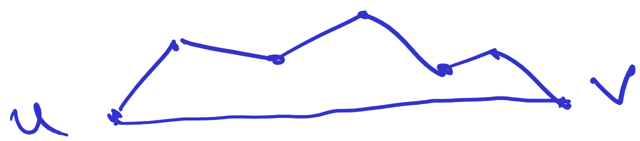
Proof: Let T' be an optimal Steiner tree. Double the edges of T'. Call the resultant multigraph T''.



The degree of each vertex of T'' is even. T'' has an Euler circuit H. $c(H) = 2 \cdot c(T')$.



$c(H') \leq c(H)$



$$c(T) \leq c(H'') \leq c(H') \leq c(H) = 2 \cdot c(T') = 2 \cdot OPT$$

$\blacksquare$

$$\Rightarrow c(T) \leq 2 \cdot OPT.$$

Proof of claim 2: Assume that $P \neq NP$.

Def: A Hamitonian cycle of a graph G is a cycle that contains every vertex of G exactly once.

(Hamiltonian Cycle Problem). Given an undirected graph, does there exist a Hamitonian cycle?

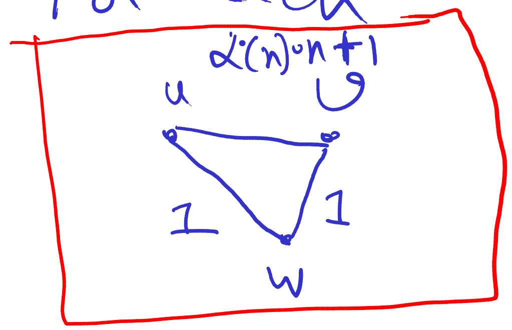
Fact: Hamiltonian Cycle Problem (HC) is NP-complete. Since $P \neq NP$,

(S1) There does not exist a polynomial time algorithm for HC.

Towards a contradiction, assume that there exists a polytime $\alpha(n)$-approximation algorithm A for TSP. We now describe a polynomial time algo B for HC:

Input to B: An undirected graph $G=(V,E)$.

- Let G' be the complete graph on V. For each pair of vertices $\{u,v\}$,



$$c(\{u,v\}) = \begin{cases} 1 & \text{if } \{u,v\} \in E. \\ n\cdot\alpha(n)+1 & \text{otherwise.} \end{cases}$$

[If G has a Hamitonian cycle H, then G' has a TSP tour of cost n.

If G does not have a Hamiltonian cycle, then each TSP tour of G' has cost at least $n\cdot\alpha(n)+1$.]

- B runs A on (G', c) to obtain a TSP tour of cost K.
- If $K \leq \alpha(n)\cdot n$, return "Yes".

  otherwise return "No".

B solves the HC problem in polynomial time, which is a contradiction to (SI). ▨

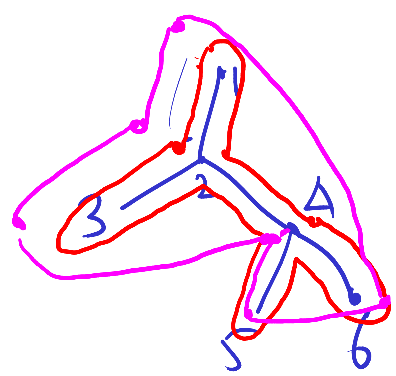
Metric TSP

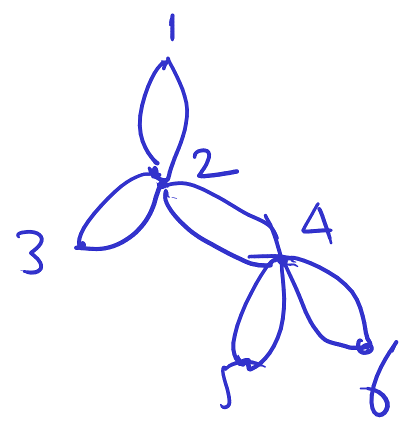
Input: Complete undirected graph $G=(V,E)$, $c: E \to \mathbb{R}^{\geq 0}$ satisfying triangle inequality.

Algorithm:

- Compute an MST T of T.
- Double the edges of T. Let H be an Euler circuit of the resultant multigraph.
- Retain the first occurrences of each vertex. Let the resultant circuit be H'.
- Return H'.

T:





1 2 3 2 4 5 4 6 4 2 1

1 2 3 X 4 5 X 6 X X 1

$$c(H') \leq c(H) \quad (\text{By } \Delta\text{-inequality})$$
$$= 2 \cdot c(T). \quad \text{——— (1)}$$

Claim 4: $c(T) \leq OPT$.

Proof: Let H be an optimum TSP tour. Remove any edge e from H. $H \setminus \{e\}$ is a path containing each vertex exactly once. In particular, $H \setminus \{e\}$ is a spanning tree.

$$\overset{OPT}{\overset{\shortparallel}{c(H)}} \geq c(H \setminus \{e\}) \geq c(T). \quad \blacksquare$$

By claim (4) and Eq^n (1) we have that
$$c(H') \leq 2 \cdot OPT. \quad \blacksquare$$

Christofides algorithm

Fact: "Given undirected weighted G, find a minimum cost perfect matching": this problem has a poly-time algorithm. $\hookrightarrow$ matches all vertices.

Let $G=(V,E)$ be a complete undirected weighted graph. Edge weights satisfy triangle inequality. OPT : cost of an optimal TSP tour. Let $A \subseteq V$. $|A|$ is even. Let M be the cost of a minimum weight perfect matching of the subgraph of G induced by A.

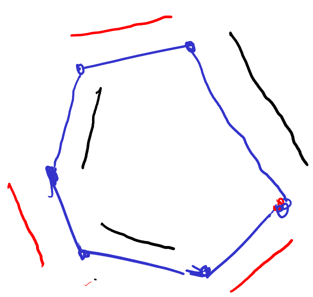
<u>Claim 1:</u> $M \leq \frac{1}{2} \cdot OPT$.

<u>Proof:</u> Let C be an optimal TSP tour of G.

$C = v_1 v_2 \cdots v_n v_1$. Let C' be the cycle obtained by dropping the vertices outside of A from C. C' is a TSP tour of the subgraph induced by A.

$$C' = u_1 u_2 \cdots u_\ell .$$

$w(C') \leq w(C)$. (Follows from the metric property w)



M_1

M_2

$$2M \le w(M_1) + w(M_2) = w(C') \le w(C) = OPT$$

$$\Rightarrow M \le \frac{1}{2} \cdot OPT.$$

∎

$$OPT \ge \begin{cases} 2 \cdot \text{cost of a min cost perfect matching of an induced subgraph} \\ \text{cost of an MST.} \end{cases}$$

Algorithm:

1. Compute an MST T of G.

2. Let $A \subseteq V$ be the set of vertices whose degree in T is odd.

$\Rightarrow |A|$ is even. $\Big[\displaystyle\sum_{v \in V} \deg_T(v) = 2|E|$

$$\Rightarrow \sum_{v \in \bar{A}} \deg_T(v) + \sum_{v \in A} \deg_T(v) = 2|E|$$

$$\Rightarrow \sum_{v \in A} \deg_T(v) = \underbrace{2|E| - \sum_{v \in \bar{A}} \deg_T(v)}_{\text{even}} \Big]$$

$\Rightarrow |A| = \text{even} \Big]$.

3. Compute a min.cost perfect matching M_1 of the subgraph of G induced by A.

4. Consider the multi-graph $H := M_1 \cup T$ (retain multiple copies of edges in $M_1 \cap T$). Each vertex has even degree in H. Compute an Euler circuit H' of H. Bypass repeated occurrences of vertices to obtain a valid TSP tour H''.

5. Return H''.

$$w(H'') \leq w(H) = w(M_1) + w(T)$$

$$\leq \underbrace{\tfrac{1}{2} \cdot OPT}_{\text{from Claim}} + OPT = \tfrac{3}{2} \cdot OPT.$$

$\blacksquare$

Linear Programming :

$$\text{max } 3x_1 + 7x_2 \leftarrow \text{objective fn.}$$

L : s.t.

$$\left.\begin{array}{l} 4x_1 + 6x_2 \le 7 \\ 2x_1 + 7x_2 \le 8 \end{array}\right\} \text{Linear constraints}$$

val(L)

$\hookrightarrow$ DLP_1: Given an $\underbrace{LP}_{\text{max}}$ L and a rational number r, is $val(L) \ge r$?

DLP_2: Given a set of linear constraints, can they be simultaneously satisfied.

Not satisfiable $\leftarrow$ $\boxed{\begin{array}{l} x \le 7 \\ -x \le -8 \end{array}}$

<u>Exercise</u>: Assume that DLP_2 is in NP. Can you prove that DLP_1 is in NP?