Layering

n vertices
m edges
≥0

Weighted vertex cover: Input: An undirected graph G=(V, E) and a cost function c:V->R. Task: Output a vertex cover with minimum cost.

A layering algorithm consists of the following steps:

(1) Identifying a set T of instances which are easy / trivial to solve with factor α.

(2) Given an instance I, express I as "sum" of simple instances. $I_1 + I_2 + \cdots + I_t$ where $I_j \in T$

(3) Trivially solve each I_j to obtain sol^n S_j. Combine those solutions to construct a solution S for I.

$$c(S) \le c(S_1) + c(S_2) + \cdots + c(S_t) \quad \text{---} (1)$$

(4) $$OPT \ge OPT_1 + \cdots + \underbrace{OPT_j}_{\text{(optimum of } S_j} + \cdots + OPT_t \quad \text{---} (3)$$

$$c(S_j) \le \alpha \cdot OPT_j \quad \text{---} (2)$$

$$c(S) \le \sum_j c(S_j) \le \alpha \cdot \sum OPT_j \le \alpha \cdot OPT.$$

Step 1: Regular graphs (i.e., degrees of all the vertices are equal to d). $c(v) = 1 \ \forall v$.

Algorithm: Output V.

Let an optimum cover be A. The number of edges covered by vertices in A is at most $|A| \cdot d$.

$$|A| \cdot d \geq m = \frac{d \cdot |V|}{2}$$

$$\Rightarrow |V| \leq 2|A|.$$

Definition: A cost function c is degree-weighted if there exits a constant K s.t. for all vertices v,

$$c(v) = K \cdot \deg(v).$$

Let A be an optimum cover.

Then m = set of edges covered by A

$$\leq \sum_{v \in A} \deg(v).$$

$$c(A) = K \cdot \sum_{v \in A} \deg(v) \geq K \cdot m = K \sum_{v \in V} \deg(v)/2$$

$$= \sum_{v \in V} c(v)/2 = \frac{c(V)}{2}.$$

Algorithm:

- $S \leftarrow \phi$.
- Include all vertices with cost 0 to S. Remove them from the graph.
- Remove all isolated vertices from the graph.

Find the largest k s.t. $\forall v$, $k \cdot \deg(v) \le c(v)$

while the graph is non-empty

- $k \leftarrow \min_{v} \dfrac{c(v)}{\deg(v)}$.

$k_1(v) = k \cdot \deg(v)$

- $c(v) \leftarrow \underbrace{c(v) - k \cdot \deg(v)}_{\text{residual cost function}}$

$c(v) - k_1(v) - k_2(v) = 0$.

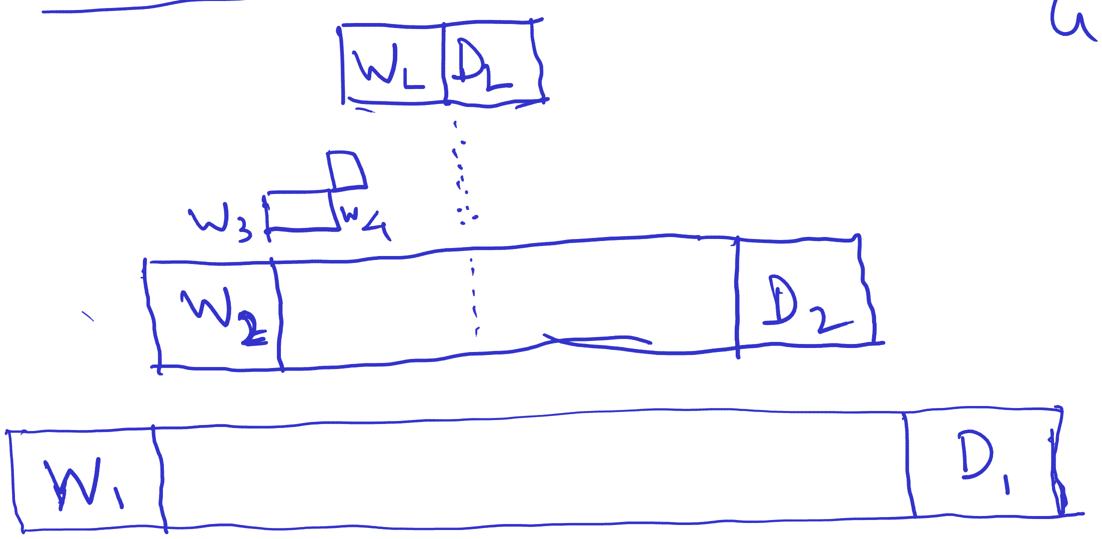
$\Rightarrow c(v) = k_1(v) + k_2(v)$.

$W_1 = \{v : c(v) = k_1(v)\}$.

W_2

- $W = \{v : c(v) = 0\}$; $S \leftarrow S \cup W$.
- Remove W from the graph.
- Let D be the set of isolated vertices in the current graph; Remove D from the graph.

Return S.



G_L k_L.

G_2 k_2

G_1 $k_1(\cdot)$

$S = W_1 \cup W_2 \cup \cdots \cup W_L$. $c(S) = \sum_{i=1}^{L} c(W_i) = \sum_{i=1}^{L}\left(\sum_{j=1}^{i} k_j(W_i)\right)$

S^*: An optimum vertex cover

$$= \sum_{j=1}^{L}\left(\sum_{i=j}^{L} k_j(W_i)\right) = \sum_{j=1}^{L}\left(k_j(W_j \cup W_{j+1} \cup \cdots \cup W_L)\right)$$

Viewed as a set of vertices.

$$\le \sum_{j=1}^{L} k_j(G_j) \le 2 \cdot \sum_{j=1}^{L} k_j(\text{an optimum vertex cover of } (G_j, k_j))$$

$$\le 2 \cdot \sum_{j=1}^{L} k_j(S^* \cap G_j)$$

$S^* \cap G_j$: Vertex cover for G_j.

- check.

$$= 2 \cdot \sum_{j=1}^{L}\left(\sum_{i=j}^{L}\left(k_j(S^* \cap W_i) + k_j(S^* \cap D_i)\right)\right)$$

$$= 2 \cdot \sum_{i=1}^{L}\left(\sum_{j=1}^{i}\left(k_j(S^* \cap W_i) + k_j(S^* \cap D_i)\right)\right)$$

$$\ge 2 \cdot \sum_{i=1}^{L}\left[c(S^* \cap W_i) + c(S^* \cap D_i)\right]$$

typo!
should be
$\le$

$$= 2 \cdot c(S^*) = 2 \cdot OPT.$$