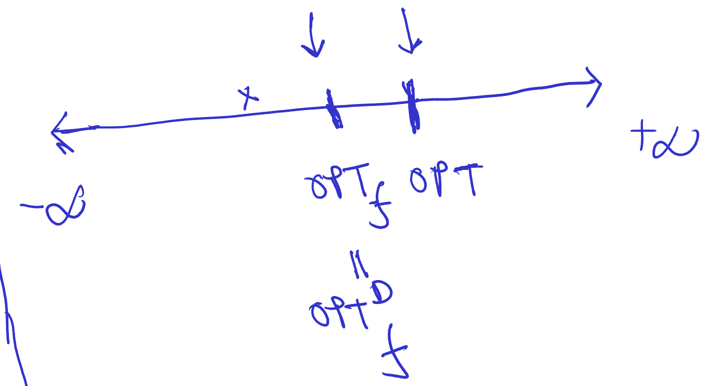
Dual fitting

A technique to analyze approximation algorithms using the theory of linear programming and duality.

1. Write the ILP formulation, relax & compute its dual.
2. Use the algorithm to set the dual variables, such that the value of the dual objective = cost of the computed sol^n.
3. Divide each value of the dual variable by a suitable factor c. Value of the dual objective also gets divided by c.
4. Show that the setting of the new dual variables is dual feasible.



Cost of the computed sol^n = Dual obj$_1$

$= c \cdot$ Dual obj$_2$ $\leq c \cdot OPT_f \leq c \cdot OPT$

Set-Cover LP relaxation:

Input: $\mathcal{U} = \{1, \dots, n\}$, $\mathcal{S}$ of subsets $\mathcal{U}$. $c: \mathcal{S} \to \mathbb{R}^+$

Primal (L'):

minimize $\sum_{S \in \mathcal{S}} c(S) \cdot x_S$

subject to: $\sum_{S \in \mathcal{S}:\, i \in S} x_S \geq 1 \quad \forall i \in \mathcal{U}$

Dual var. y_i for $i \in \mathcal{U}$

$-x_S \geq -1 \quad \forall$ set $S \in \mathcal{S}$, $(x_S \leq 1)$

redundant

$x_S \geq 0$

Dual (L):

maximization $\sum_{i=1}^{n} y_i = |T|$

$T = \{i : y_i = 1\}$

s.t. $\sum_{i \in S} y_i \leq c(S) \quad \forall S \in \mathcal{S}$

$\boxed{OPT \geq T}$

$y_i \geq 0 \quad \forall i \in \mathcal{U}$.

$y_i \in \mathbb{Z}$

new elements that S covers.

$e_1\ e_2\ e_3\ \boxed{e_i}\ e_{n-1}\ e_n$.

Assume that the set picked in the iteration in which e_i got covered is S. $\text{Price}(e_i) = \frac{c(S)}{\#\text{ of "new" elements that } S \text{ covers}}$.

Consider the following setting of dual variables?

$$y_i := \text{price}(i).$$

Cost of the computed solution $= \sum_{i=1}^{n} \text{price}(i).$

$$\boxed{\sum_{i \in S} y_i \le c(S)}$$

$$\sum_{i \in S} \text{Price}(i).$$

Let $e_1\, e_2 \,.\, e_j\,.\, e_k$ be the elements of S in the order in which they are covered in the algorithm.

In the iteration in which e_j gets covered, set S is available to the algorithm.

$$\text{Price}(e_j) \le \frac{c(S)}{k-j+1}.$$

$$\sum_{i \in S} \text{Price}(i) = \sum_{j=1}^{k} \text{Price}(e_j) \le c(S) \cdot \sum_{j=1}^{k} \frac{1}{k-j+1}$$

$$= c(S) \cdot \left[1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{k}\right] = c(S) \cdot H(k)$$

$$y_i := \frac{\text{Price}(i)}{H(n)}.$$

$$\sum_{i \in S} \frac{\text{Price}(i)}{H(n)} = \frac{1}{H(n)} \cdot \sum_{i \in S} \text{Price}(i)$$

$$\le \frac{c(S) \cdot H(k)}{H(n)} \le c(S).$$

$\Rightarrow$ Approximation factor $\leq H_n$.

Utility:

1. Lower bounding scheme: The dual objective value for a dual feasible point. OPT_f

$$\frac{c(\text{computed sol}^n)}{\text{OPT}} \leq \frac{c(\text{computed sol}^n)}{\text{OPT}_f} \leq H_n.$$

2. General applicability.