

Online paging

Theorem 1: Competitive ratio of any deterministic online paging algorithm is at least $k \rightarrow$ cache size.

Proof: Let A be a deterministic online paging algorithm. Consider pages $1, \dots, k+1$. Let δ_i be the page in $\{1, \dots, k+1\}$ that is missing in A 's cache after processing $\delta_1, \dots, \delta_{i-1}$. The number of misses that A suffers is n (i.e., length of the request stream).

Consider the algorithm (not online) FITF Furtherst-in-the-future. Whenever a page not in the cache is requested, the page whose next request is latest. Observe

that FITF incurs at most one miss in any k consecutive requests. Assume that, in a step, page i is evicted from the cache, and page l is brought in. Before this step, the content of the cache is $\{1, \dots, k+1\} \setminus \{l\}$.

All this k pages will be requested before i . After i is evicted, there $k-1$ pages in the cache which will be requested before i . So, no miss for next $k-1$ steps.

\therefore The total number of misses by FIFO is at most $\lceil \frac{n}{k} \rceil$.

\therefore The competitive ratio of our algorithm is at least k .

$$C(A, \delta) = n, \quad \text{OPT}(\delta) \leq \lceil \frac{n}{k} \rceil \leq \frac{n}{k} + 1.$$

Let α be the competitive ratio of A . Hence, there exists a constant c st.

$$C(A, \phi) \leq \alpha \cdot \text{OPT}(\phi) + c \quad \forall \text{ seq. } \phi.$$

$$\Rightarrow C(A, \phi) \leq \alpha \cdot \text{OPT}(\delta) + c$$

$$n = C(A, \phi) \leq \alpha \cdot \left(\frac{n}{k} + 1 \right) + c$$

$$\Rightarrow n(1 - \frac{\alpha}{k}) \leq c. \quad - (1)$$

If $\alpha < k$, $1 - \frac{\alpha}{k} > 0$ and (1) is false for large enough n .

$$\Rightarrow \alpha \geq k.$$

Conservative algorithms: An algorithm is conservative

if for every request sequence with at most k distinct requests, the number of misses is at most k . Example: LRU, FIFO.

Exercise: Check that LRU & FIFO are conservative algorithms.

Theorem 2: Any conservative algorithm is k -competitive.

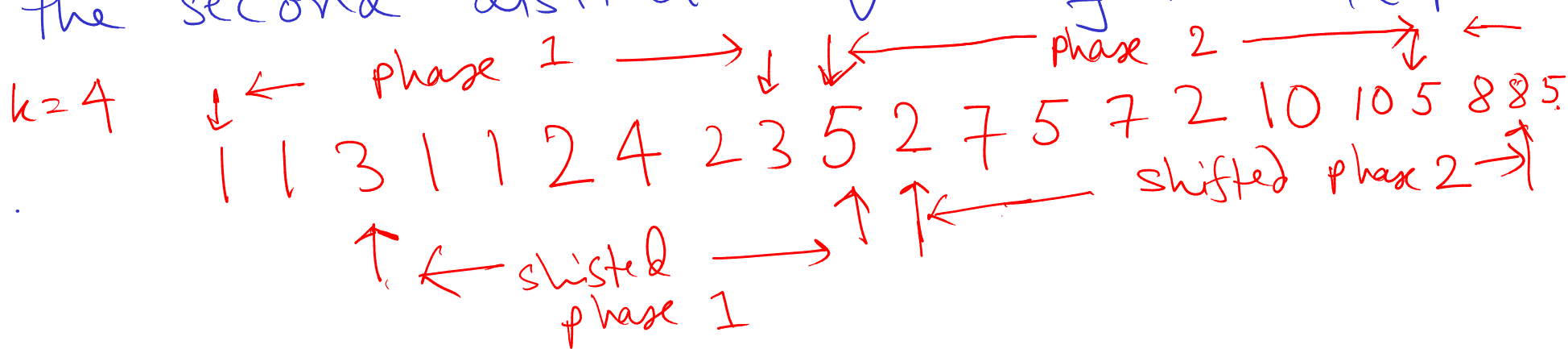
Proof: Let $\delta = \delta_1, \delta_2, \dots$ be some request sequence. We will partition δ into phases.

Phase 1: $\delta_1, \dots, \delta_i$ where i is the maximum index for which $\delta_1, \dots, \delta_i$ has ^{at most} k distinct pages.

Phase 2: A maximal ^{seq. of} consecutive requests starting from δ_{i+1} which has ^{at most} k distinct pages.

In each phase, the algorithm incurs at most k misses (from the defⁿ of conservative algorithm).

Let $a = a_1 a_2 \dots$ be an optimum answer sequence. A **shifted phase** starts from the second distinct request of a phase and ends just before the second distinct request of the next phase.



We claim that a must incur 1 miss in every shifted phase.

Let p be the first request in this page. Hence, p is present in the cache in the beginning of this shifted phase. There are $k-1$ ^{distinct} requests in this phase which are different from p . Also, the first request of the next phase is different from p . Thus, in this shifted phase we will see k requests which are different from p . Since the number of pages different from p that are in the cache in the beginning of the shifted phase is $k-1$, therefore a incurs at least one miss in this shifted phase.

Thus in any sequence with l phases,

→ a conservative algorithm incurs at most lk misses

→ Any answer sequence incurs at least $l-1$ shifted phases.

⇒ Competitive ratio of a conservative algorithm is at most k . □