An asymptotic PTAS

Theorem 1: $\forall\, \varepsilon \in (0, \frac{1}{2}]$, there is an algorithm A that runs in time polynomial in n, and finds a packing using at most $(1+2\varepsilon)\,OPT+1$ bins.

Proof strategy: $\Rightarrow \forall\, \varepsilon \in (0,1]$, $\exists A \cdots$ $(1+\varepsilon)\,OPT+1$ bins.

1. Design a polytime exact algorithm for instances where all sizes are at least ε, and number of distinct sizes is at most some fixed constant K.

2. Design a polytime $(1+\varepsilon)$-approximate solution for instances where all sizes are at least ε.

3. Prove Theorem 1.

Step 1: All sizes $\geq \varepsilon$, No. of distinct sizes $\leq K$.

Each bin can hold at most $\lfloor \frac{1}{\varepsilon} \rfloor$ items.

Number of possible bin configurations.

$\{s_1, \cdots, s_K\}$: set of all possible sizes. Let a bin contain N_i items of size s_i, $i = 1, \cdots, k$. $\sum_{i=1}^{k} N_i \le \lfloor \frac{1}{\varepsilon} \rfloor := M$

$$\underbrace{* * * \cdots *}_{N_1} \mid \underbrace{* * \cdots *}_{N_2} \mid \cdots \mid \underbrace{* \cdots *}_{N_k} \mid \underbrace{* \cdots *}_{M - \sum_{i=1}^{k} N_i}$$

Number of such patterns is $\binom{M+K}{K}$.

$\Rightarrow$ Number of bin configurations $\le \binom{M+K}{K} := R$

Number of bins used is at most n.

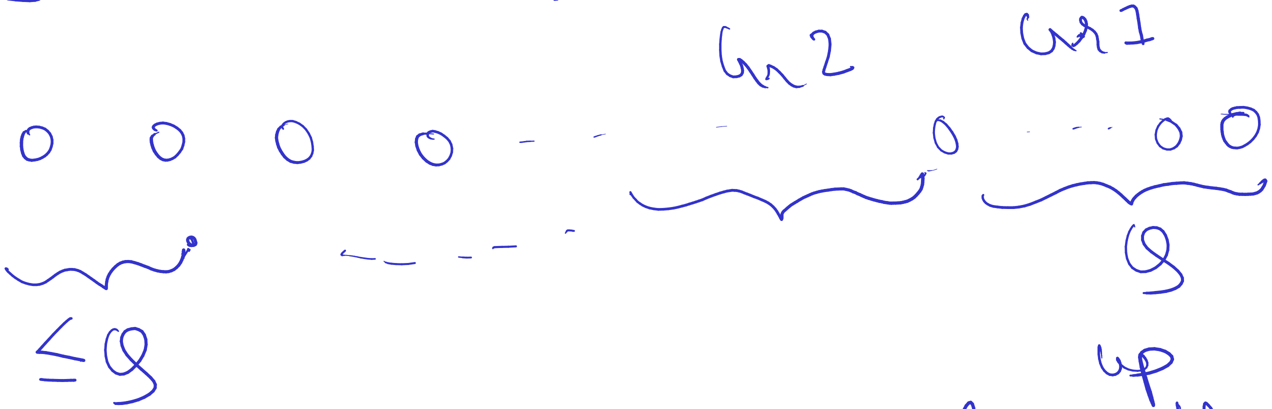
Number of joint configurations of $\le n$ bins is at most $\binom{n+R}{R} = O(n^R)$.

Brute force over all possible joint configurations.

Let I be the given instance.

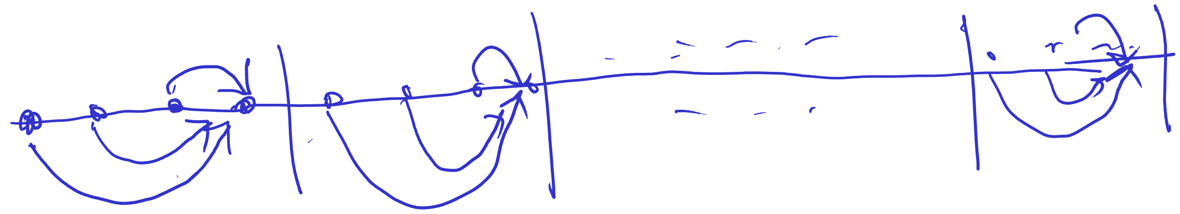
<u>Step 2</u>: All sizes $\ge \varepsilon$. Sort the items in non-descending order of sizes. Partition them into $K = \lceil \frac{1}{\varepsilon^2} \rceil$

groups, each having equal number (g) of items, except possibly one (first) group.



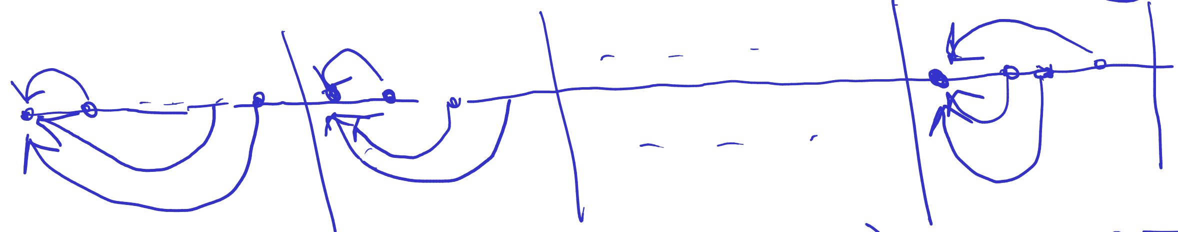
Construct instance J by rounding up the size of each item to the size of a largest item in its group.

J

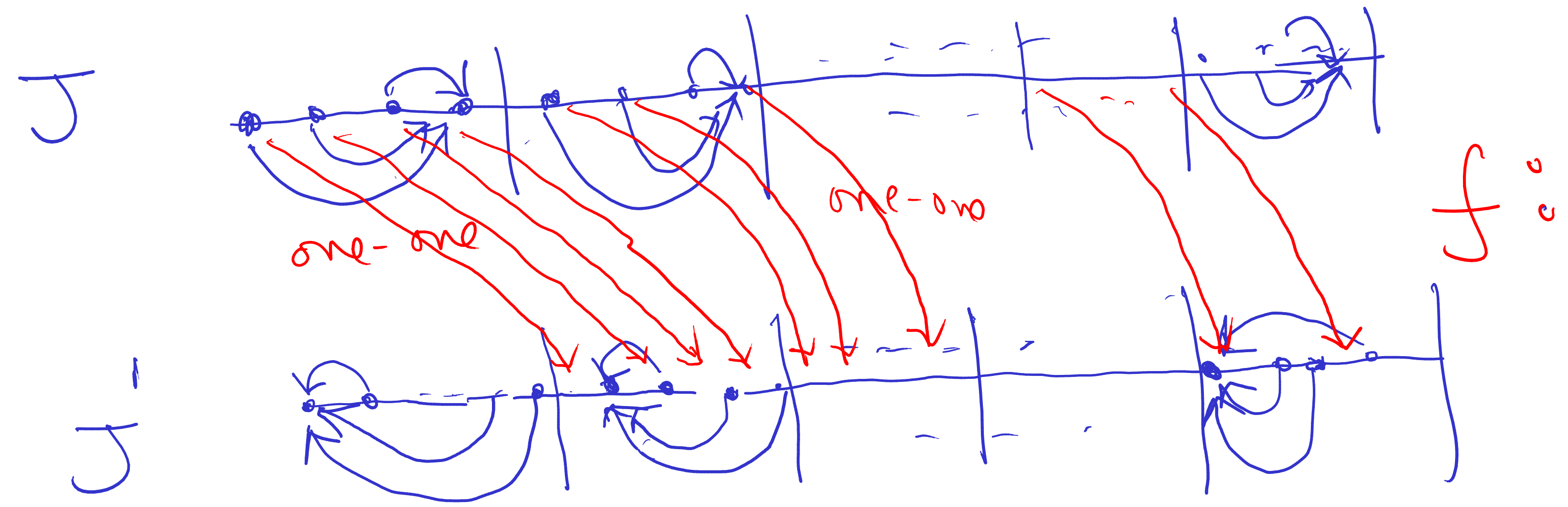


Use the algorithm for step 1 to get a packing. Since "I is dominated by J sizewise", the packing found is also a feasible packing for I.

J'



$$OPT(J') \le OPT(I) \le OPT(J)$$



obs. $size_J(i) \le size_{J'}(f(i))$.

A feasible packing for J' is also a feasible packing for J for all but the last Q items.

$$Q \le \frac{n}{\lceil \frac{1}{\varepsilon^2} \rceil} \le \lfloor \varepsilon^2 n \rfloor.$$

A packing for J: Use an optimal packing of J' to pack all but the last Q elements. Then put each of the last Q elements in a separate bin.

$$OPT(J) \le \underbrace{OPT(J')}_{} + \underbrace{Q}_{(\le \varepsilon^2 n)} \le OPT(I) + \varepsilon^2 n \quad (1)$$

$$OPT(I) \ge \sum size_I(i) \ge \varepsilon n \quad (2)$$

From ①, $OPT(J) \leq OPT(I)(1+\varepsilon)$. (From ②)

Step 3:

- Pack all items of size $\geq \varepsilon$ using step 2, using at most $(1+\varepsilon)\cdot OPT$ many bins.
- Use First-Fit to pack the items with size $< \varepsilon$. Open new bins if you cannot pack an item in any existing bin.

Analysis: Case I: No addition bin is used. Then the no. of bins is at most $(1+\varepsilon)\cdot OPT$.

Case 2: Additional bins are used. Let m bins be used in total. This implies that at least $m-1$ bins are full upto the capacity of $1-\varepsilon$.

$$OPT \geq \text{Sum of sizes}$$
$$\geq (m-1)(1-\varepsilon)$$
$$\Rightarrow m \leq \frac{OPT}{1-\varepsilon} + 1 \leq (1+2\varepsilon)\cdot OPT + 1. \quad \left[\frac{1}{1-\varepsilon} \leq 1+2\varepsilon \text{ if } \varepsilon \leq \frac{1}{2}\right]$$

(#) $\exists$ LP-rounding based algo that returns a sol^n with cost $OPT + O(\log^2 OPT)$.