Randomized rounding

$U = [n]$,
$S_1, S_2, \dots, S_m$
Cost of S_i is $c(S_i)$

$$\text{minimize} \sum_{i=1}^{m} c(S_i) \cdot x_i$$

s.t.

$$\sum_{j : i \in S_j} x_j \geq 1 \qquad \forall i \in U = \{1, \dots, n\}$$

$$x_i \geq 0 \qquad \forall i = 1, \dots, m$$

$$-x_i \geq -1$$

$$x_i \in \mathbb{Z}$$

Relax (replace by $x_i \in \mathbb{R}$)

Let L be the relaxation of the set-cover ILP.

Algo A1:

1. Solve L to obtain an optimum feasible point $\bar{x}^* = (x_1^*, x_2^*, \dots, x_m^*)$.

$x_1^* = 0.23$

2. Rounding: $C \leftarrow \phi$.
   For $i = 1$ to m
   $C \leftarrow C \cup \{S_i\}$ w.p. x_i^* independently.

③ Return $\mathcal{C}$.

$\mathbb{E}[c(\mathcal{C})]$.

For every $i = 1, \ldots, m$,

$$I_i = \begin{cases} 1 & \text{if } S_i \in \mathcal{C} \\ 0 & \text{otherwise.} \end{cases}$$

$$c(\mathcal{C}) = \sum_{i=1}^{m} I_i \cdot c(S_i) \quad \text{— (1)}$$

Fact 1 (Linearity of expectation): For any two random variables X and Y, $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.

$$\mathbb{E}[c(\mathcal{C})] = \mathbb{E}\left[\sum_{i=1}^{m} I_i \cdot c(S_i)\right] \quad (\text{By } ①)$$

$$= \sum_{i=1}^{m} \mathbb{E}[I_i \cdot c(S_i)] \quad (\text{By fact } ①)$$

$$I_i \cdot c(S_i) = \begin{cases} c(S_i) & \text{w.p. } x_i^* \\ 0 & \text{w.p. } 1 - x_i^* \end{cases}$$

$$\mathbb{E}[I_i \cdot c(S_i)] = x_i^* \cdot c(S_i)$$

$$\mathbb{E}[c(\mathcal{C})] = \sum_{i=1}^{m} x_i^* \cdot c(S_i) = \mathrm{Val}(L)$$

$$\leq OPT$$

$\Pr[\mathcal{C} \text{ covers } j] = ?$ $\qquad j \in \{1, \dots, n\}$.

Let E_j be the event that $\mathcal{C}$ <u>does not</u> cover j.

$\forall S_i$ that contains j, $\Pr[S_i \notin \mathcal{C}] = (1 - x_i^*)$

$$\Pr[E_j] = \prod_{i: j \in S_i} \Pr[S_i \notin \mathcal{C}] = \prod_{i: j \in S_i} (1 - x_i^*)$$

$$\leq \prod_{i: j \in S_i} e^{-x_i^*} = e^{-\sum_{i: j \in S_i} x_i^*} \leq \frac{1}{e}.$$

$\boxed{1 - z \leq e^{-z} \text{ For all real numbers } z}$

Algorithm $\mathcal{A}$:

- Run $\mathcal{A}_1$ for t steps independently.
- Let the collections returned by $\mathcal{C}_1, \dots, \mathcal{C}_t$
- Return $\mathcal{C} := \mathcal{C}_1 \cup \mathcal{C}_2 \cup \dots \cup \mathcal{C}_t$.

$$c(\mathcal{C}) \leq \sum_{i=1}^{t} c(\mathcal{C}_i). \qquad \mathbb{E}[c(\mathcal{C})] \leq \mathbb{E}\left[\sum_{i=1}^{t} c(\mathcal{C}_i)\right]$$

$$= \sum_{i=1}^{t} \mathbb{E}[c(\mathcal{C}_i)] \le t \cdot OPT.$$

(By Fact 1)

For $j \in \mathcal{U}$,

$$\Pr[C \text{ does not cover } j] \le \frac{1}{e^t}.$$

$$\Pr[\exists j \in \mathcal{U}, \text{ s.t. } C \text{ does not cover } j]$$

$$= \Pr\left[\bigcup_{j=1}^{n} C \text{ does not cover } j\right]$$

$$\le \sum_{j=1}^{n} \Pr[C \text{ does not cover } j]$$

$$\le \frac{n}{e^t}.$$

$$\left[e^t = e^{2\log_e n} = \left(e^{\log_e n}\right)^2 \right]$$

Choose $t := 2\log_e n$.

Then $\Pr[\underbrace{\underbrace{\exists j \text{ s.t. } c \text{ does not cover } j}_{C \text{ is not a set-cover}}}_{\text{event } F_1}] \le \frac{n}{\left(e^{\log_e n}\right)^2} = \frac{1}{n}$

$\le \frac{1}{4}$

(Assuming $n \ge 4$)

$$\mathbb{E}[c(\mathcal{C})] \le 2\log_e n \cdot OPT.$$

$$\Pr[\underbrace{c(\mathcal{C}) > 8\log_e n \cdot OPT}_{\text{event } F_2}] \le \frac{2\log_e n \cdot OPT}{8\log_e n \cdot OPT} = \frac{1}{4}$$

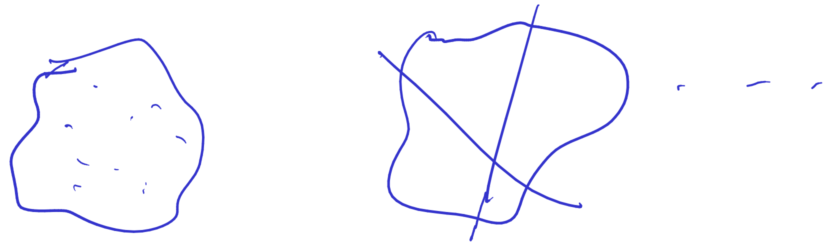
$$\Pr[F_1 \cup F_2] \le \Pr[F_1] + \Pr[F_2] = \frac{1}{2}.$$

With probability at least $\frac{1}{2}$, A returns a valid set-cover of cost at most $8\log_e n \cdot OPT$.

<u>Q3</u> (G, w, R). A: algo. for problem P.

Let $r \in R$. $S := \{r\}$, $R' := R \setminus \{r\}$.

Run A on (G, S, R'). Say the output is T.



OPT_{ST} OPT_P

$OPT_{ST} \le OPT_P$.