Seventh Week: Lower bounds using Kolmogorov complexity

Computational Complexity

We know that the Kolmogorov complexity $K(x)$, of a string x of n bits is no more than $n + c$, where c is a constant, independent of n. We also know that there is at least one string of n bits, say $x*$, such that $K(x*) \geq n$. In other words, we have a string that is not so (computationally) compressible. [Establish these two facts using fundamental arguments.]

In addition, we can also show that $K(x)$ is not computable. Further, we can develop an alternative proof of the undecidability of the halting problem using the fact that $K(x)$ is not computable. [Exercises.]

For the palindromes problem, assume that the machine M recognizes palindromes and therefore in particular palindromes of the kind $x0^nx^r$ where $|x| = n$ and x^r is the reverse of the string x. [We eventually will use the existence of a string $x*$ of length n such that $K(x*) \geq n$ in order to show that deciding palindromes has time complexity $\Omega(n^2)$ on a 1-tape deterministic Turing machine.] If we look at the crossing sequences then there are small ones, and a ith one, $n \leq i \leq 2n$, called S_i, such that its length is $m \leq T(x)/n$; here, $T(x)$ is the total time complexity for recognizing $x0^nx^r$ as a palindrome. We can now observe that x is the only string which when appended with 0^{i-n} will *respect* the crossing sequence, as otherwise, if $y \neq x$ too did so, then $y0^nx^r$ would also be accepted by M as a palindrome. So, the crossing sequence S_i uniquely identifies x. It is easy to see that $S_i's$ description in bits would be smaller than a constant times $T(x)/n$ plus $2\log_2 n$ (for storing n and i). This description can be used by another machines M' to generate x. Since the bit length of S_i is dominated by $c_M T(x)/n + 2\log_2 n$, $K(x)$ cannot exceed this bit length by more than a constant $c_{M'}$. However, there is a n-bit string $x*$ for which $K(x*) \geq n$. Thus, $n \leq K(x*) \leq c_m T(x*)/n + 2\log_2 n + c_{M'}$, whence the quadratic lower bound.