Linear speedup

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Last week we considered space compression. Now we consider a similar result for time complexity. For an (intuitively) super-linear time-bounded machine, arbitrary constant factor (with respect to the input size) speedups are possible.

1 Linear speedup

We simulate a $k > 1$ tapes, $T(n)$-time bounded TM compressing m cells into 1 to get the same language recognized in $cT(n)$ time, provided $T(n)/n \to \infty$ and $n \to \infty$, when we choose m such that $mc \geq 16$. That is, to get speed up $1/c$ times, we require $c \geq 16/m$. [This result is established by using that fact about $T(n)$ above that for any d, there is an n_d such that $n \geq n_d$ implies $T(n)/n \geq d$ or (equivalently) $n \leq T(n)/d$. That is, the special limit $T(n)/n$ tends to infinity.]

The steps include initial reading of n cells of M_1's input and converting that n symbol string into an $\lceil n/m \rceil$-symbol string on M_2's own machine. Then, M_2 simulates $T(n)$ steps of M_1 in its $8\left\lceil \frac{T(n)}{m} \right\rceil$ moves.

Exercise 1 *Show that M_2 can indeed simulate $T(n)$ steps of M_1 as above.*

So, M_2 requires following number of moves

$$n+\lceil n/m \rceil+8\cdot\lceil T(n)/m \rceil \leq n+(n/m+1)+(8T(n)/m)+8 \leq n+n/m+8T(n)/m+9$$

Since $inf_{n\to\infty} T(n)/n = \infty$, for any constant d (however large), there is an n_d so that for $n \geq n_d$, we have $T(n)/n \geq d$ or, equivalently, $n \leq T(n)/d$. Also, putting $n \geq 9$ (thus $n+9, \leq 2n$) we have the above upper bound on the number of steps of M_2 as $T(n)[\frac{2}{d} + \frac{1}{md} + \frac{8}{m}]$ for $n \geq n_d$. [Note that $n + 9 < 2n$, so $n + 9 \leq 2n \leq 2T(n)/d$.] [These terms therefore come from respectively, $n + 9$, n/m and $8T(n)/m$.] Now choose $m \geq 6/c$ and $d = m/4 + 1/8$, we have the simulation time of M_2 as less than $cT(n)$. [Note that $\frac{2}{d} + \frac{1}{md}$ is the same as $8/m$ if $d = \frac{m}{4} + \frac{1}{8}$. And $8/m$ is less than $c/2$!]

This is the main idea in the proof of Theorem 12.3 in [HU79].

Exercise 2 *Establish and interpret the corollary following this theorem in [HU79].*

Exercise 3 *What happens in contrast when $T(n)$ is a constant multiple of n? (See Theorem 12.4 and its corollary in [HU79].)*

Exercise 4 *Extend these results to nondeterministics Turing machines.*

Exercise 5 *Study Theorems 12.5 and 12.6 in [HU79].*