Test 2: March 24, 2023: 11 am

Computational Geometry: LTP 3-0-0: Credits 3. Marks: 100
Time: 100 minutes
CS60064
Open book and notes

1. Argue why an ϵ-net is not necessarily an ϵ-sample (ϵ-approximation)? Does the converse hold? Why? Answer after defining ϵ-nets and ϵ-samples (ϵ-approximations). [5+5+5 marks]

2. Given four points in the plane which are not co-circular, with no three points collinear, show that these four points cannot be shattered by circular ranges. You may use the properties of the dual triangulation of the Voronoi diagram of these four points. [10 marks]

3. Given 30 points in plane, no four of which are co-circular and no three of which are collinear, estimate an upper bound on the number m of subsets S of these 30 points, where each such set S is a subset of the 30 points inside some circle. [15 marks]

4. Consider the definition of configurations called it racquets $\tau = (\sigma, e)$ where σ is a trapezoid and e is a handle or segment adjacent to σ in the trapedoidal decomposition $\Pi(R)$ for any subset R of a given set N of line segments in the plane.

   Show that these configurations have *bounded degree*. [7 marks]

   Show that this bounded degree property of such configurations leads to the fact that the number of such racquet configurations is proportional to the size of the trapezoidal decomposition $\Pi(R)$ of line segments for any set $R \subseteq N$. [8 marks]

5. Computing the Voronoi diagram of n sites of the input set N of points in the plane is easily done incrementally, tail recursively. So, we compute the Voronoi diagram $G(N^{i+1})$ of the first $i+1$ sites from the Voronoi diagram $G(N^i)$ of the first i sites by inserting the site S_{i+1} in the incremental step and modifying vertices and edges of the Voronoi diagram suitably, removing some structures (vertices and edges) and adding new structures. Provided we know any vertex p of $G(N^i)$ in the Voronoi region of S_{i+1} in $G(N^{i+1})$, show how we can compute $G(N^{i+1})$. [15 marks]

6. During the addition of $S = S_{i+1}$, we now need to update conflict information accordingly. If the conflict pointer from some halfspace $I \in N \setminus\ N^{i+1}$ points to a vertex r in $cap(S_{i+i})$, then we need to find another vertex in $p \in H(N^{i+1})$ in conflict with I.

   Observe that some newly created vertex of $H(N^{i+l})$, contained in S_{i+1}, must conflict with I.

   We set $j = i+1$. The part of $H(N^i)$ cut off by S_{i+1} is called $cap(S_{i+1})$. If the conflict vertex r for a certain $I \in N \setminus N^{i+1}$ is in $cap(S_{i+1})$ then we need to search for a new conflict vertex for I, which must be a newly created vertex in $H(N_{i+1})$.

   The search will see old vertices in the cap but they get deleted anyway and thus we focus only on the number $k(N^j, S^j, I)$ of *newly created vertices* in $H(N^j)$ in conflict with I, and adjacent to S_j.

   The sum of the number of all such newly created vertices over all adjacent $S \in N^j$, is no more than d times the number $k(N^j, I)$ of vertices in $H(N^j)$ in conflict with I. Explain why so. [5 marks]

   The expected cost $\frac{1}{j}\Sigma_{S\in N^j} k(N^j, S, I)$ of search in the $j(= i+1)$th step is thus at most $\frac{d \times k(N^j, I)}{j}$, just for the the halfspace I.

   But this needs to be summed up over all $I \in N \setminus N^j$, giving the upper bound

$$\frac{d}{j}\Sigma_{I\in N\setminus N^j} k(N^j, I) \tag{1}$$

Looking from another angle, for $j+1 = i+2$, the expected value by definition of $k(N^j, S_{j+1})$ is

$$E[k(N^j, S_{j+1}))] = \frac{1}{n-j} \Sigma_{I \in N \setminus N^j} k(N^j, I) \tag{2}$$

Why? [5 marks]

Now we can write Equation 1 using Equation 2 above as

$$\frac{d}{j}(n-j)E[k(N^j, S_{j+1})]$$

7. Consider the computation of the common intersection of n halfplanes $S_1, S_2, ..., S_n$, in a random sequence. In adding Sj, we need to simply consider the number $m(S, N^j)$ of newly created vertices in $H(N^j)$. Why? [7 marks]

   The expected cost of this jth step is thus $\frac{1}{j}\Sigma_{S \in N^j} m(S, N^j)$.

   For each $j < n$, let $e(j)$ denote the expected size (number of vertices) of $\Pi^0(N^j)$, assuming that the N^j is a random sample (subset) of N of size j.

   Why is the expected number of newly created vertices in the jth random addition, conditional on a fixed N^j, bounded by $\frac{d}{j}$ times the number of active vertices over N^j (i.e., the ones in $H(N^j)$)? [8 marks]

   Why is the expected cost of the jth step $\frac{1}{j}\Sigma_{S \in N^j} m(S, N^j)$ at most $\frac{d}{j}$ times $e(j)$? [5 marks]