Approximation and Online Algorithms: CS60023: Autumn 2018: S P Pal *Copyrights reserved*

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Heuristics for vertex cover for graph $G(E, V)$

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- The second one chooses both vertices of all edges in a maximal matching S.

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- This yields a vertex cover that is certainly at most twice the size of the minimum vertex cover.

The NP-completeness reduction

- The NP-complete reduction from 3-SAT to vertex cover constructs a graph $G_f(V_f, E_f)$ for each 3-SAT CNF formula f, such that f is satisfiable if and only if G_f has a vertex cover of size exactly $k = 2|V_f|/3$.

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- The edges form triangles for each clause; three edges between pairs of literals in each clause.
- More edges join inconsistent pairs of literals across the clause triangles, like x_i with x_i'.
- Note that the minimum vertex cover must have size at least $2/3|V_f|$.

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- The function U assigns a value 'T' or 'F' to each variable of f, thereby assigning true or false value to each literal in each clause.
- In G_f, we have one vertex for each literal of each clause, totalling to $3m$ vertices in all.

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- So, any cross edge e is covered at least at one of its ends by some vertex in C.
- If both ends are covered then we choose any one, say vertex v arbitrarily.

The NP-completeness reduction

- If $x(v)$ is the boolean variable in the 3-CNF formula corresponding to the vertex v, then we assign $x(v) = f$ if the literal corresponding to the vertex v is the uncomplemented literal for boolean variable $x(v)$, and we assign $x(v) = t$, otherwise.

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- So, the literal at the other end of the cross edge e is assigned 'T' in the truth assignment with $x(v)$ assigned as above.
- In this way, the truth value 'F' is assigned for exactly two literals of the formula in every clause, corrresponding to the two vertices of the vertex cover C of the corresponding triangle.

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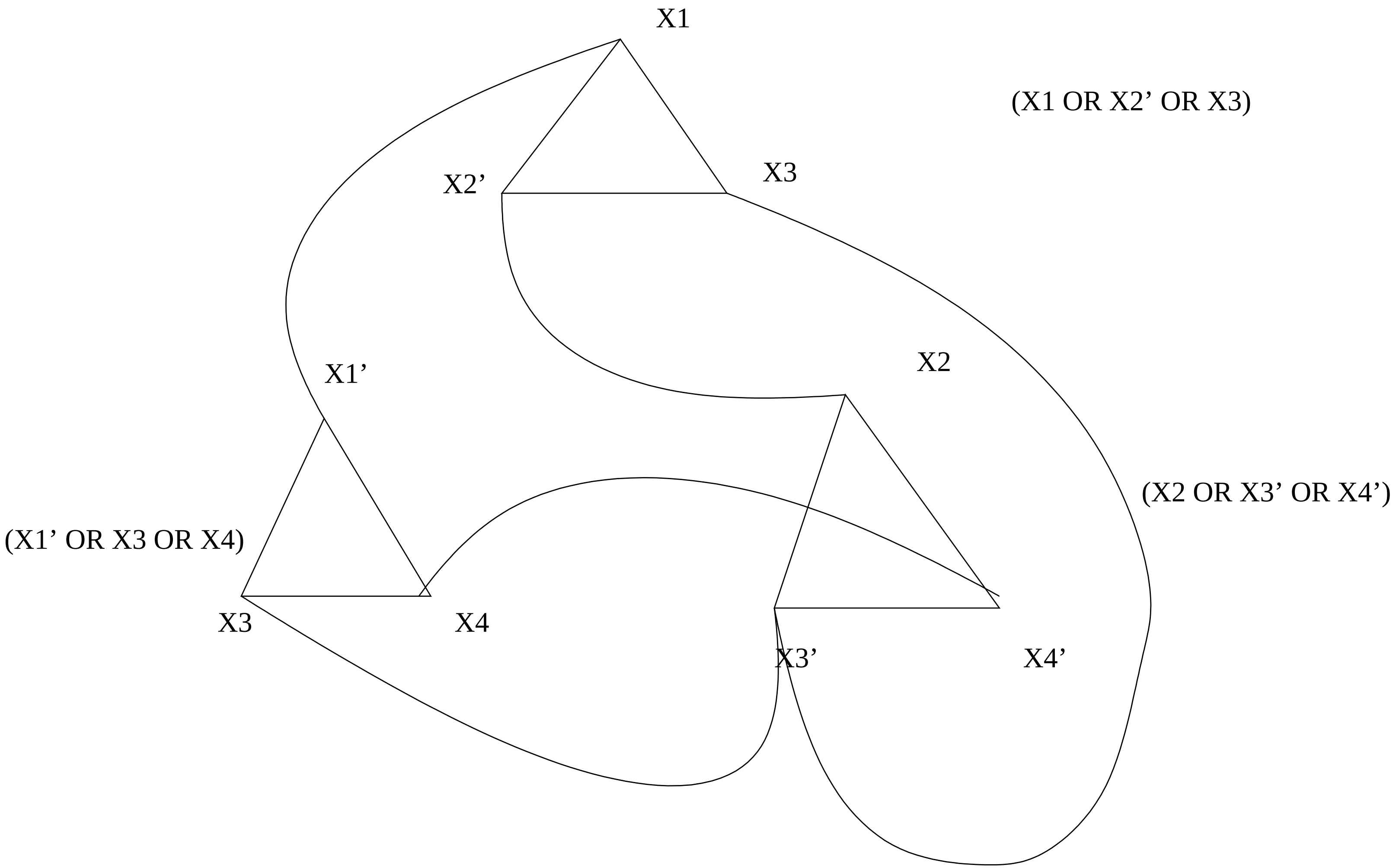
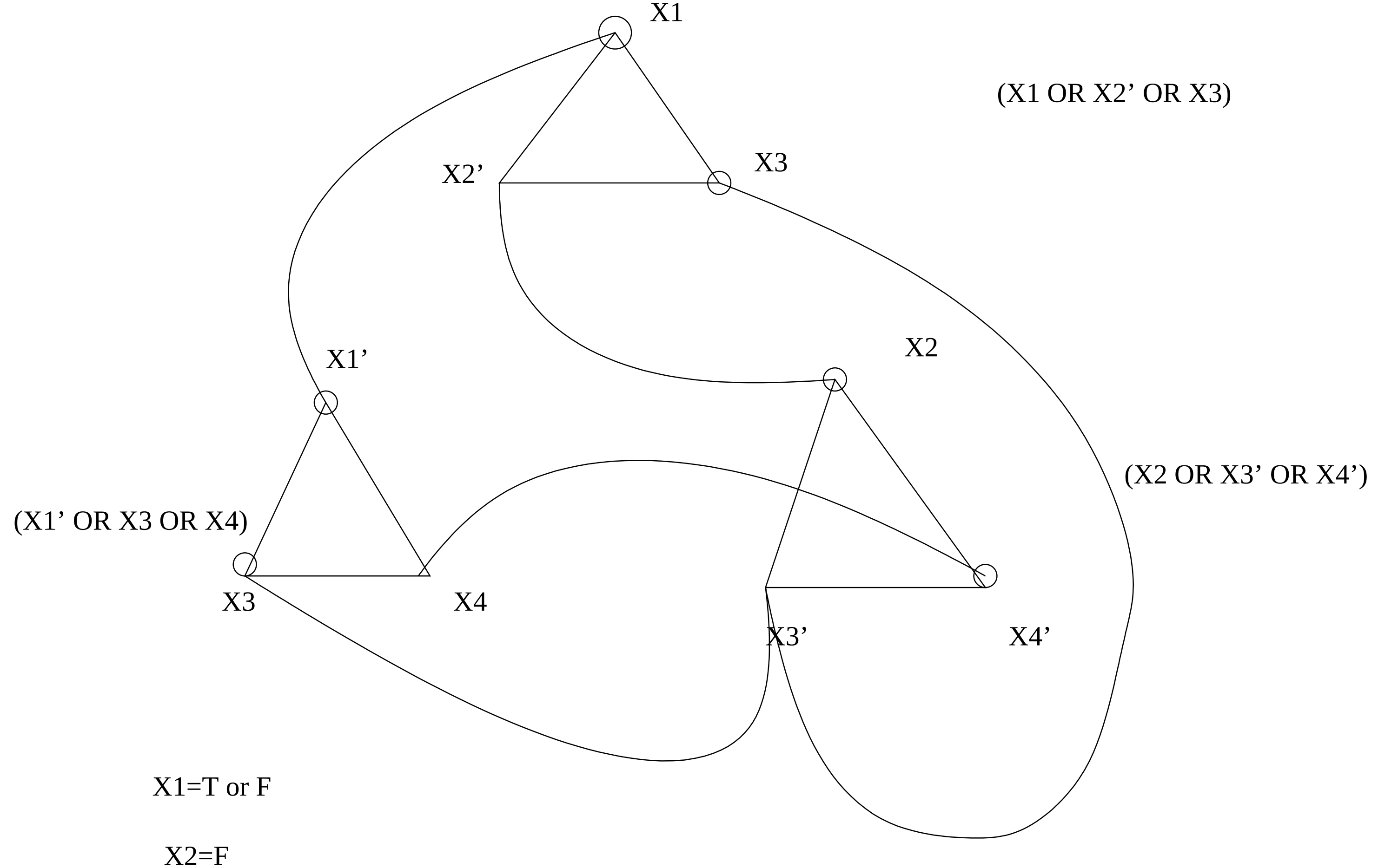
- Therefore the truth assignment thus designed must be a satisfying truth assignment for the boolean 3-CNF formula.

The construction of G_f from 3-CNF formula f



One minimum vertex cover



X1=T or F

X2=F

X3=F X4=T

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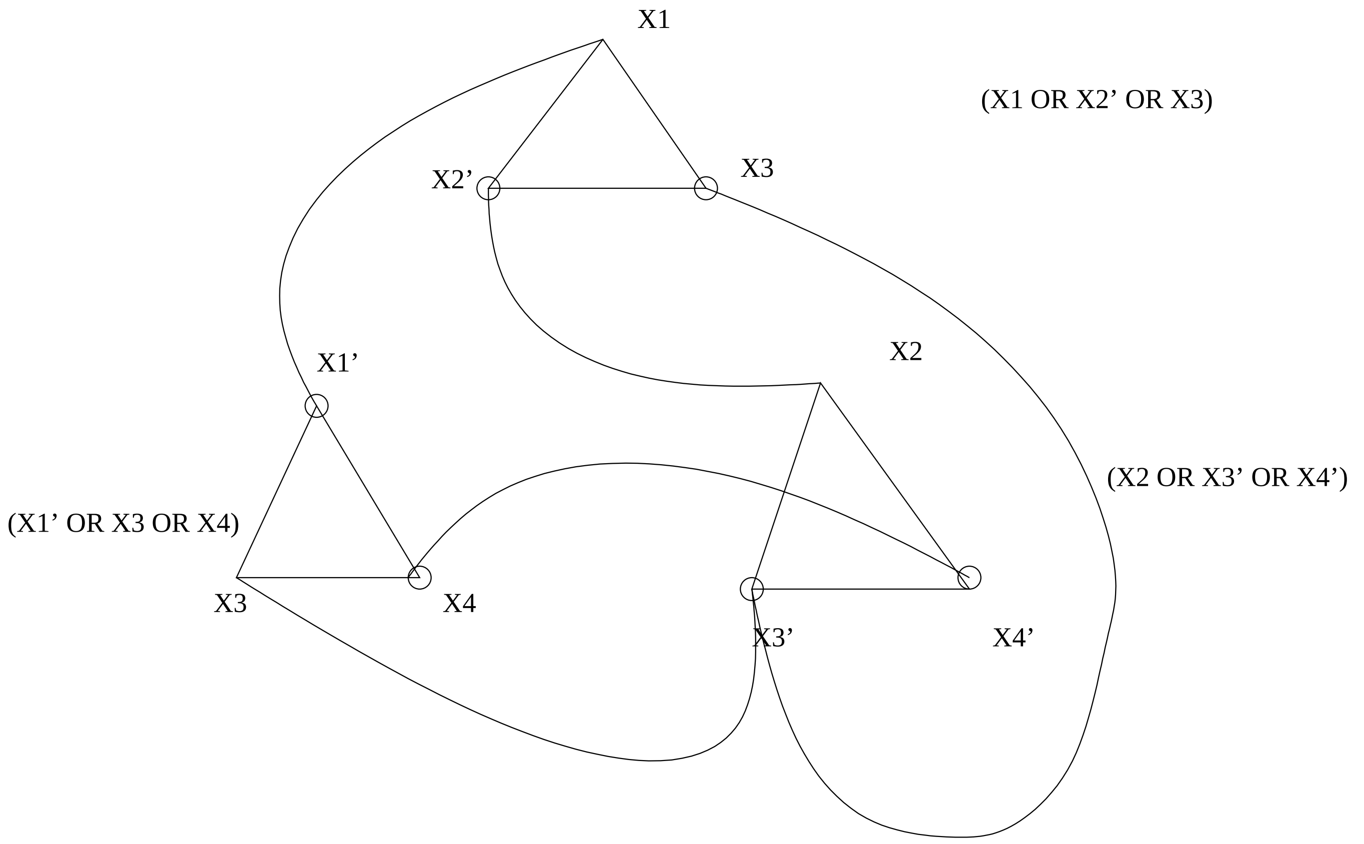

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- The vertices getting the colour which colours the smallest number of vertices are at most $\lfloor n/3 \rfloor$ in number.

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- The similarity with vertex cover for graphs is that vertices covered edges for graphs, whereas vertices cover regions (triangles) of the polygon for the art gallery problem.
- Savings in the number of vertex guards is possible if we note that several guards see common regions, beyond their own triangles. The art gallery problem of minimizing vertex guards is NP-hard.

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- In order to view this as a 'vertex covering' problem, consider the set $\mathcal{S}$ of k sets as corresponding to the set V of all vertices in the graph $G(V, E)$, where the set E has one element $\mathcal{S}_u \in E$ for each element $u \in U$ and each set $S \subseteq U$ such that $u \in S \in \mathcal{S}_u$, and such that $\mathcal{S}_u \subseteq \mathcal{S}$.

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- The n elements in U correspond to the (hyper)edges $\mathcal{S}_u$ of the (hyper)graph $G(V, E)$.

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- We show that such a cover C can be found in polynomial time with ratio bound $O(\log |U|)$, that is, $|C| = O(|C^*| \log |U|)$.

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- We show that such a cover C can be found in polynomial time with ratio bound $O(\log |U|)$, that is, $|C| = O(|C^*| \log |U|)$.
- Surprisingly, a simple heuristic works; we choose sets $S \in \mathcal{S}$ in decreasing order of the number of new elements covered, until all elements of U are covered. The sets thus selected constitute the collection $C \subseteq \mathcal{S}$.

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- If the $i - 1$ sets selected so far are S_1, S_2, ...,S_{i-1}, then we have already assigned some ‘prices’ to the elements of these sets.

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- Each element is charged with a *price* only once; let the *price* assigned to an element $x \in U$ be c_x.

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- We now define a quantity $\sum_{S \in \mathcal{C}^*} \sum_{x \in S} c_x$ for an (unknown) optimal set cover $\mathcal{C}^*$. [We will succeed in showing that this quantity is indeed an upper bound on $|\mathcal{C}|$.]

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- Here, $H(n) = O(\log n)$ is the harmonic sum $\sum_{1 \leq i \leq n} \frac{1}{i}$

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- The reason is that the sets in the optimal cover $\mathcal{C}^*$ might intersect. So, an element of x may be counted several times on the right hand side of the inequality whereas each element is counted exactly once on the left hand side.
- We assume (as shown in Cormen et al. pages 1036-37), that $\sum_{x \in S} c_x \leq H(|S|)|S \in \mathcal{S}$
- Here, $H(n) = O(\log n)$ is the harmonic sum $\sum_{1 \leq i \leq n} \frac{1}{i}$
- We can now see that $|\mathcal{C}| \leq \sum_{S \in \mathcal{C}^*} H(|S|) \leq |\mathcal{C}^*|.H(max\{|S| : S \in \mathcal{S}\})$

The general weighted set cover problem

- In this case, each set $S \subseteq U$, has a positive and rational weight $c(S)$. Here, U is the universal set of n elements and the collection of sets $\mathcal{S} = \{S_1, S_2, \cdots, S_k\}$.

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- We now show how the ratio approximation factor of $H(n)$ is attained.
- The greedy selection rule for the next set S is similar to the rule in the unweighted set cover heuristic.

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- In each greedy step, we select that set S whose cost effectiveness is minimum; for each element $e \in S$, we say $price(e) = \alpha$.
- Now, let $e_1, e_2, \cdots, e_n$ be the sequence in which the selected sets covered the n elements.

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- We establish this bound below; first we show how to use this bound.
- We know that summing $price(e)$ over all $e \in U$ gives us the sum of weights of sets in the set cover computed by our greedy algorithm.
- This is clearly $H(n) \times OPT$, by the use of the above upper bound for $price(e)$, which we now proceed to establish below.

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- Since all the n elements can be covered with OPT cost by the already chosen sets of the optimal solution C^* and the leftover sets of the optimal solution, it is also possible for the leftover sets of the optimal solution to cover the remaining elements with cost at most OPT.

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- Since all the n elements can be covered with OPT cost by the already chosen sets of the optimal solution C^* and the leftover sets of the optimal solution, it is also possible for the leftover sets of the optimal solution to cover the remaining elements with cost at most OPT.
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- Therefore, our algorithm will greedily select some set covering the kth element with at most

$$price(e_k) \leq \frac{OPT}{|U \setminus C|} \leq \frac{OPT}{n-k+1}$$

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  $given \sum_{j=1}^{n} a_{ij} x_j \geq b_i, i = 1, \cdots, m [Ax \geq b]$
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- Here, A is an $m \times n$ matrix, b is an $m \times 1$ matrix, and x and c are an $n \times 1$ matrices. Note that b is a lower bound on Ax, whereas we cannot indefinitely inflate x since we wish to minimize $c^T x$.

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- If we wish to address the question of membership in P or NP, it helps to formulate decision versions of the *linear programming* problem.
- Instead of computing x^*, we may ask whether $z^* = c^T x^*$ is at most α, where α is a real number.
- Note that we do not know z^* when we are given the decision version instance, denoted by matrices A, b, c and α. Nevertheless, we pose the decision version question "whether $z^* \leq \alpha$".

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- Such an a is a feasible (possibly non-optimal) solution which is a 'witness' that this is a 'yes' instance.
- The moment we know such a 'witness' a, we set $d = c^T a$, and we can easily check whether $Aa \geq b$ and $c^T a \leq \alpha$, confirming and verifying that z^* is also at most α, that is, $z^* \leq c^T a = d \leq \alpha$.

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- We simply can check efficiently given such a certificate a, that the given instance is indeed a 'yes' instance.
- Is this decision question also in the class co-NP? We will soon answer this question after we define what is known as the *dual* problem of a given (*primal*) linear program.

Linear Programming: Duality

- $$minimize \sum_{j=1}^{n} c_j x_j [c^T x]$$

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- $$maxmize \sum_{i=1}^{m} b_i y_i [b^T y]$$

  $$given \sum_{i=1}^{m} a_{ij} y_i \leq c_j, j = 1, \cdots, n [A^T y \leq c]$$

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- Symmetrically, the upper bounds in the constraints of the dual program define the objective function in the primal program.
- Observe further that the ‘variables’ in $y \geq 0$, in the dual linear program are multipliers of the lower bounds in b of the primal linear program.
- Even though we maximize the objective function in the dual, which is the dot or inner product of b with the weight- or price- or the variables- vector y, we are well guarded by the upper bounds in $c \geq A^T y$.

Linear Programming: Weak duality

- So, we are ensured that the coefficients of each primal variable x_i, in all the m inequalities of the primal, when weighted by the m multipliers or variables in y of the dual, do not exceed the corresponding cost c_i of the primal.

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- With this intuition, we now proceed to formally establish the 'weak duality' result below.

Linear Programming: Weak duality

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$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i [c^T x \geq b^T y]$$

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- $$\sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) y_i \geq \sum_{i=1}^{m} b_i y_i [y^T A x \geq b^T y]$$

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- This is easy to show using a similar argument applied to suitable feasible solutions of the dual linear program that have lower bounded objective function values.
- Using such solutions of the dual as ‘certificates’ or ‘witnesses’, ‘yes’ instances of this new problem can be shown to be checkable in polynomial time.

Dual fitting analysis technique for the greedy set cover

- The problem of minimum set cover is as follows.
  $minimize \sum_{S\in\mathcal{S}} c(S)x_S$ subject to
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- The *dual* linear program is
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- We also know that the cost of any feasible solution to the dual linear program is no more than OPT_f, which in turn is no more than OPT.
- The optimal costs of the primal and dual linear programs are both OPT_f.
- When we choose the next element $e_i \in S = \{e_1, e_2, \cdots, e_k\}$ of the k elements of a set S in the greedy set cover heuristic, the $price(e_i)$ is no more than $\frac{c(S)}{k-i+1}$, as we now demonstrate.

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- Clearly, $price(e_j) = \frac{c(S)}{k-i+1} \leq \frac{c(S)}{k-j+1}$, $i \leq j \leq k$.
- Otherwise, some other set includes e_i with lower cost effectivity, such that $price(e_i) \leq \frac{c(S)}{k-i+1}$, as per the greedy algorithm.

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- Now setting the variable y_e of the dual linear program for each $e \in U$ to $\frac{price(e)}{H(n)}$, we observe that

$$y_{e_i} \leq \frac{1}{H(n)} \cdot \frac{c(S)}{k - i + 1}$$

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- So,

$$\sum_{i=1}^{k} y_{e_i} \leq \frac{c(S)}{H(n)}.(\frac{1}{k} + \frac{1}{k-1} + \cdots + \frac{1}{1}) = \frac{H(k)}{H(n)}.c(S) \leq c(S)$$

- So, the constraints in the dual linear program are satisfied establishing the feasibility of the solution with y_e values as assigned above. Now we further observe that

$$\sum_{e \in U} price(e) = H(n)(\sum_{e \in U} y_e) \leq H(n).OPT_f \leq H(n).OPT$$

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- The set of constraints represents the intersection of half-spaces, which is a convex region of multi-dimensional space, called the *feasible* region.
- Optima of linear objective functions like $c^T x$ can occur only at vertices of this convex feasible region.

Formulation with weights for vertices

- Being more precise, the problem we define is as follows. *Given an* $m \times n$ *matrix* A*, and vectors* $b \in \mathcal{R}^m$ *and* $c \in \mathcal{R}^n$*, find a vector* $x \in \mathcal{R}^n$ *solving the optimization problem min*$\{c^T x$ *such that* $x \geq 0$ *and* $Ax \geq b\}$.

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- Each vertex i has a positive weight w_i. We say that the weight of a set of vertices is the sum of weights of its vertices.
- We wish to compute a vertex cover with at most twice the optimal weight in polynomial time.

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- The minimum weighted vertex cover will minimize

$$\sum_{i \in V} w_i x_i$$

such that

$$x_i + x_j \geq 1, (i, j) \in E$$

and

$$x_i \in \{0, 1\}, i \in V$$

Discrete Integer Linear Program

- We can rewrite the problem formally as

$$Ax \geq 1$$

$$1 \geq x \geq 0$$

where the integer 0-1 matrix A has one row for each edge and one column for each vertex and $A[e, i] = 1$ whenever vertex i is in edge e and 0, otherwise.

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- We have reduced the optimization version of the minimum weighted vertex cover problem to the linear programming problem where we require the solutions (for x_i) to be from $\{0, 1\}$.

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- This problem is called *0-1 integer programming*, and due to the NP-hardness of the vertex cover problem, this problem is also NP-hard.
- Why is the *decision version* of this 0-1 integer programming problem also in the class NP?

Linear Programming relaxation for the ILP

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- If $x_i^* \geq \frac{1}{2}$, only then we include i in S. Why do we get a vertex cover?
- This way we get an approximate vertex cover, whose total weight will now be shown to be at most twice the optimal.

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- So, we have $w(S) \leq 2w_{LP} \leq 2w(S^*)$.