Advanced graph theory: Tutorial October 18, 2024: CS60047 Autumn 2024

Time: 100 minutes

1. In Dirac's proof for the sufficient condition that a connected undirected graph has a Hamiltonian circuit if $\delta(G) \geq \frac{n}{2}$, show how you can make the condition weaker.

2. For graph H, let $ex(n, H)$ denote the largest possible number of edges in a graph G with n vertices, $n \geq |H|$, where $|H|$ is the number of vertices of H and H is not a subgraph of G. Let $\chi(H)$ be the vertex chromatic number of the undirected connected graph H. Let $t_{r-1}(n)$ denote the number of edges of the n-vertex Turan graph $T_{r-1}(n)$. The Turan graph $T_{r-1}(n)$ is the unique complete $(r-1)$-partite graph with $n \geq r-1$ vertices whose $r-1$ partite sets differ in size by at most 1.

   (i) Is H a subgraph of the Turan graph $T_{\chi(H)-1}(n)$ for any $n \in \mathcal{N}$, where $\mathcal{N}$ is the set of natural numbers? Why? (4 marks)

   (ii) Is $H \subseteq K^s_{\chi(H)}$ for some sufficiently large s? Why? (4 marks)

   (iii) How is $ex(n, K^s_{\chi(H)})$ related to $t_{\chi(H)-1}(n)$? State the relationship without proof. (4 marks)

   (iv) Count the number of distinct K_{r-1}'s in $T_{r-1}(n)$ where n is a multiple of $r-1$. (3 marks)

   (v) Show that $ex(n, H) \geq t_{\chi(H)-1}(n)$.

   (vi) Show that $ex(n, H) \leq ex(n, K^s_{\chi(H)})$

   (vii) Show that $ex(n, H) < t_{\chi(H)-1}(n) + \epsilon n^2$, for a sufficiently large n.

   (viii) Prove Corollary 9.12 in Pach and Agarwal.

3. Show that a graph G is perfect if and only if for every induced subgraph H of G, we have $\alpha(H)\omega(H) \geq |H|$.