# Advanced graph theory: Tutorial 3: CS60047 Autumn 2024

Time: 70 minutes; Maximum marks: 100

August 23, 2024

1. The easier part of Kuratowski's theorem is to show that the presence of homeomorphs of $K_5$ or $K_{3,3}$ as subgraphs would make a graph non-planar. Prove this by (i) showing that $K_5$ and $K_{3,3}$ are non-planar, and (ii) the presence of a homeomorph of a non-planar graph causes non-planarity.

   [A graph $G$ is a homeomorph of another graph $H$ if $G$ can be obtained by repeatedly adding degree-2 vertices $w$ by deleting edge $\{u, v\}$, and adding edges $\{u, w\}$ and $\{w, v\}$. Note that $H$ is planar if and only if its homeomorph $G$ is planar.]

   [This amounts to showing the necessary condition that homeomorphs of none of the two Kuratowski's graphs can appear as subgraphs in a planar graph.]

   [The tougher part of Kuratwoski's theorem is to show that a graph is planar if it does not have subgraphs homeomorphic to the any of the two Kuratowski graphs.]

2. Show that every 2-connected 3-regular graph has a 1-factor.

3. Use Hall's theorem to show that $\beta(G) \leq \alpha'(G)$ in a bipartite graph $G(X \cup Y, E)$.

   (10 marks)

   [Show that for any minimum cardinality vertex cover $C$ in $G$, we can demonstrate a matching $M$ of the same size. The minimum vertex cover $C$ can span both the two partites of the bipartite graph $G$. Such a partition of the vertex cover $C$ into two parts, say $C \cap X$ and $C \cap Y$, would create two disjoint sub-bipartite graphs $G(C \cap X, Y \setminus C)$ and $G(C \cap Y, X \setminus C)$, where $N(S)$ denotes the neighbourhood set of vertices of $S$ in $G$. Apply Hall's theorem to both these sub-bipartite graphs.]

4. Minimum $s - t$ separating vertex cuts are called *$s - t$ clots.* The cardinality $\kappa(s, t)$ of an *$s - t$ clot* is called the *$s - t$ clot number*.

   Show that the vertex connectivity $\kappa(G)$ of a graph $G$ is the minimum of all non-adjacent pair clot numbers in $G$.

   A *thread* between $s$ and $t$ of a graph $G$ is a set of paths between $s$ and $t$ which have pairwise no vertices common except for $s$ and $t$. The number of paths in a thread is called the *ply number* of the thread. The maximum ply number of a thread between $s$ and $t$ is called the *$s - t$ ply number* and written as $p(s, t)$. The *ply number* $p(G)$ of $G$ is the minimum of $p(s, t)$ over all non-adjacent pairs $s$ and $t$ in $G$.

   Explain the vertex version of the theorem of Menger in terms of the *ply* and *clot* numbers, $p(s, t)$ and $\kappa(s, t)$, respectively. [5 marks]

   A graph $G$ is called *$k$-connected* if $k \leq \kappa(G)$. Show that a graph is $k$-connected iff any two of its vertices are joined by a thread of ply number $k$, as per your above statement of Menger's theorem in terms of ply and clot numbers [3 marks].

5. Show that $K_5$ and the Petersen graph are not planar graphs.

## 6. An easy application for bipartite graphs

Use Hall's theorem show that a bipartite graph $G(V, E)$ has a perfect matching if and only if $|N(S)| \geq |S|$ for every $S \subseteq V$, where $V$ is the set of all the vertices of the graph $G(V, E)$.

7. Every $k$-regular, $(k-1)$-connected graph with even order has a 1-factor.

8. If $G$ is a $k$-regular, $(k - 1)$-connected graph of even order and $G'$ is obtained from $G$ by removing any $(k - 1)$ edges then $G'$ has a 1-factor.