Advanced graph theory: Tutorial 2: CS60047 Autumn 2024

Time: 70 minutes; Maximum marks: 100

August 16, 2024

1. A connected simple planar graph with m edges, n vertices and girth g satisfies $m \leq \frac{g(n-2)}{g-2}$.

2. The *thickness* of G is the least integer k so that G has *planar partition* $[G_1, G_2, ..., G_k]$. A *planar partition* of G is a collection $\mathcal{G} = [G_1, G_2, ..., G_k]$ of edge-disjoint spanning subgraphs of G, whose union is G. Derive a lower bound for the thickness $\theta(G)$ of G in terms of the number m of edges of G, the girth g of G, and the number of vertices n of G.

3. Apply Tutte's theorem to answer the following questions. Let $G' \neq K_n$ be an n-vertex simple connected undirected graph where adding any new edge e would introduce a perfect matching in $G' + e$, given that G' has no perfect matching. If S is the "bad" set as per Tutte's theorem whereby $o(G' - S) > |S|$, then show that (i) S induces a complete subgraph in G', (ii) each connected component of $G' - S$ also introduces a complete subgraph in G', and (iii) the vertices in S are connected to all the vertices in G'.

4. Show that 3-regular graphs with no cut edges have a 1-factor.

5. Show that every regular graph of even degree has a 2-factor.

6. Show that every 3-regular graph with at most two cut edges has a 1-factor.

7. Show that every 2-connected 3-regular graph has a 1-factor.

8. For the bipartite graph $G(A \cup B, E)$, the subsets we consider are $X \subset A$, irrespective of whether the size of the neighbourhood $N(X) \subseteq B$ of X in G is equal to or greater than $|X|$. Here A is the set to be matched into B. Hall's condition requires $N(X)$ to be at least as big as X for every $X \subseteq A$, so that the whole of A may be covered by a matching. So, clearly, there are two cases, one of equality and the other of strictly

being greater.

We use induction to prove the hypothesis for matching the set A, given that the hypothesis holds for matching smaller sets that are subsets of A.

We may have (i) $N(X)$ strictly larger than X for every $X \subset A$, $X \neq \phi$, or (ii) there is at least one $A_1 \subset A$, such that $N(A_1)$ is of the same size as A_1, $A_1 \neq \phi$. These are mutually exclusive and exhaustive cases. In either case, the induction hypothesis is that there is a matching that covers any proper subset of A. Using this assumption, we need to show that there is a matching that covers A.

Work out the details of these two cases in order to show that satisfying the sufficiency condition for A implies the whole of A can be covered by a matching in G.