

Advanced graph theory: Tutorial 1: CS60047

Autumn 2024

Time: 70 minutes; Maximum marks: 100

August 02, 2024

1. Given a connected simple undirected graph G that has k triangles in a linear sequence, interspersed and separated by $k - 1$ cut edges, with a total of $3k$ vertices and $3k + (k - 1)$ edges, find $\chi(G)$, $\alpha(G)$, $\beta'(G)$, and $\omega(G)$. [10 marks]

2. Consider a connected simple undirected graph $G(V, E)$ that has the following property. For every induced subgraph H of G , $|V(H)| \leq \alpha(H)\omega(H)$. Which other graphs derived from G will also satisfy the same property? Why? [10 marks]

[Many graphs derived from G can satisfy the same property: the complement graph, the induced subgraphs of G and its complement, etc.]

3. Show that no 3-connected graph can have exactly seven edges. [15 marks]

[No 3-connected graph has exactly seven edges as $2e \geq n\kappa(G) \geq 3n$. This is because $2e \geq n\delta(G) \geq n\kappa(G) \geq 3n$. So, n must be at most $\frac{14}{3}$, i.e. at most 4. However, that can give only 6 edges.]

4. Show that in a triangle-free graph G , the number of edges is at most $\alpha(G)\beta(G)$. [10 marks]

[Each vertex in the minimum vertex cover can connect with vertices that form an independent set if there are no triangles.]

5. For a connected graph show that $2\alpha'(G) \geq \beta(G) \geq \alpha'(G)$. Characterize a family of graphs with $n > 3$ vertices where $\beta(G)$ approaches $2\alpha'(G)$ as n grows. [8+7 marks]

[Each maximum matching edge needs to be covered. Other edges can be covered by the other vertex of the matching edge. Note that K_n has a matching of size $\frac{n}{2}$ and a vertex cover of size $n - 1$.]

6. Show that in an undirected connected graph G , $\alpha(G) \geq \frac{|V|}{\Delta(G)+1}$ where $\Delta(G)$ is the maximum vertex degree. [15 marks]

[Since each of the $\alpha(G)$ vertices of the maximum independent set would have at most $\Delta(G)$ neighbours, with edges connecting to these neighbours landing only in the minimum vertex cover, we have $|V| - \alpha(G) \leq \alpha(G)\Delta(G)$, whence $\alpha(G) \geq \frac{|V|}{\Delta(G)+1}$.]

7. Show that a tree can have at most one perfect matching. [10 marks]

[A graph with two distinct perfect matchings would have cycles in the symmetric difference of the two matchings.]

8. Suppose a tree T has a perfect matching. Then show that $o(T \setminus \{v\}) = 1$ for every vertex v of T . [15 marks]

[If there were no odd components then the total number of vertices is odd. We know that $o(T \setminus \{v\}) \leq 1$. Why?]