Advanced graph theory:
Tutorial 5: November 01, 2024: CS60047
Autumn 2024

Time: 100 minutes

November 3, 2024

1. Show that every minimal non-planar graph is 2-connected. A minimal non-planar graph is a non-planar graph such that each proper subgraph is planar.

   [Hint: Show that if a minimal non-planar graph G were 0-connected (and not 1-connected), or 1-connected (but not 2-connected), then it has a planar embedding, thereby leading to contradictions.]

2. For a subset S of the vertex set of a graph G, an S-lobe of G is an induced subgraph of G whose vertex set consists of S and the vertices of a component of $G - S$. Let S be a set of two vertices x and y. Let S be a separating set of G. If G is non-planar then show that adding the edge xy to some S-lobe of G yields a non-planar graph.

   [Hint: If each S-lobe added with the edge xy is planar then show that G is planar, a contradiction.]

3. Show that if G is a graph with the fewest number of edges among all non-planar graphs without Kuratowski subgraphs, then G is 3-connected (Theorem 6.2.7 in West's book [1]).

   [Definition 6.2.3 for "Kuratowski subgraphs", "minimal non-planar graphs" in [1], and Definition 5.2.19 in [1] will be useful.]

   [Hint: The premise of this result implies the premise of Problem 1. In addition (for the sake of contradiction to the premise of this problem about not having Kuratowski subgraphs), use Problem 2 by assuming the contrapositive of this problem that G has a two-separator by not being 3-connected.]

4. Let G be a *minimal* non-planar graph with all vertices having degrees at least three.

   (a) Show that G is 3-connected. (Problem 5.37(c) [2].)

[By minimality we mean that each proper subgraph of G is planar.]

[Hint: If G has a two-vertex separator $S = \{x, y\}$ (for the sake of contradiction), then define two separated graphs which are both planar and then use their planar embeddings to get a planar embedding for G.]

(b) Show that G has a cycle with a chord. (Problem 5.37(b) [2]).

[Hint: Observe a longest path and that its first vertex has two more neighbours in the same longest path.]

(c) Also, G must be isomorphic to a K_5 or a $K_{3,3}$. (Problem 5.37(c) [2]).

[Hint: Use circuits and chords: Remove the chord of the circuit. Consider the rest of the plane and choose the circuit so that the number of regions inside be maximal. Prove that the graph has only chords outside the circuit.]

(d) Show that a graph G that is non-planar but 3-connected must have a Kuratowski subgraph. (Problem 5.37(d) [2]).

[Hint: A planar graph cannot have any Kuratowski subgraph. However, we must now show that a non-planar graph must have a Kuratowski subgraph. Use parts (a), (b) and (c) above.]

[Solution sketch: If G is not planar then G contains a minimal non-planar graph G_0. If we get rid of the vertices of degree 2, we get another minimal non-planar graph, now with vertex degrees at least three. This graph is must be either of the two Kurorowski graphs. So, G_0 must be a subdivision of one of the two Kuratowski graphs.]

References

[1] D. B. West *et al.*, *Introduction to graph theory.* Prentice hall Upper Saddle River, 2001, vol. 2.

[2] L. Lovasz, *Combinatorial Problems and Exercises.* Elsevier, 1993. [Online]. Available: https://books.google.co.in/books?id=4Yr5oQEACAAJ