CS60047 Advanced Graph Theory	2023-10-06
Lecture 12	
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This lecture develops the characterization of perfect graphs as graphs which have a complete subgraph that meets all maximum independent sets and the generation of perfect graphs by extension of a vertex and the existence of a complete subgraph in a perfect graph that meets all the maximum independent sets of the perfect graph. We go on to prove Lovász's perfect graph theorem.

12.1 Characterization of perfect graphs

Claim: *A graph is perfect if and only if every induced subgraph has an independent set (stable set) that intersects with every maximum clique in that subgraph.*

Proof(=>): Consider an arbitary perfect graph G and let H be an arbitary subgraph of G. Since H is a subgraph of a perfect graph therefore, H is also perfect. Take any $\chi(H)$ colouration of H and let S be a colour class of this colouration. Assume that S misses a maximum clique C of H. Since S misses C,

$$|C| < \chi(H) \tag{1}$$

But, $|C|$ is same as the size of the maximum clique of H. Therefore,

$$\omega(H) < \chi(H) \tag{2}$$

This implies that H is not perfect. But, this is a contradiction since we have established that H is perfect. Therefore, the initial assumption that S misses a maximum clique is false. Therefore, S meets all maximum cliques of H. Since S is a colour class and no two adjacent vertices of G can have the same colour, S is an independent set. This proves the existence of an independent set that meets all maximum cliques in every induced subgraph of G. This completes the proof of the necessary condition of the claim.

Proof(<=): Let G be any arbitrary graph and every arbitary subgraph of G has an independent set that meets all maximum cliques of the subgraph. Let H be any arbitary subgraph of G. Trivially for any graph we require atleast as many colours as the maximum clique size to colour it. Therefore,

$$\chi(H) \geq \omega(H) \tag{3}$$

Proof of the other direction of the inequality is done by induction on the number of vertices of H. Base cases of K_1 and K_2 satisfy the inequality trivially. Let S be an independent set of H that meets all maximum cliques of H. Any proper colouration of H will not require one more colour than $\chi(H-S)$. Therefore,

$$\chi(H) \leq \chi(H-S)+1 \tag{4}$$

Also, since S meets every maximum clique of H, adding S to $H-S$ will increase the size of any maximum clique by atleast 1. Therefore,

$$\omega(H) \geq \omega(H-S)+1 \tag{5}$$

Since, the subgraph $H-S$ is perfect by the inductive hypothesis we can write expression (5) as

$$\omega(H) \geq \chi(H-S)+1 \tag{6}$$

Hence, using (4) expression (5) becomes

$$\omega(H) \geq \chi(H) \tag{7}$$

Hence, by (3) and (7) we have

$$\omega(H) = \chi(H) \tag{8}$$

Since every subgraph H of G satisfies the perfect graph property we can conclude that G is perfect. This completes the proof for the sufficient condition. □

12.2 Generating perfect graphs by extension of a vertex

Construction: *A vertex x in any perfect graph G is expanded by adding a new vertex x' and connecting x' to x and all neighbours of x in G, thus obtaining the expanded graph G'.*

Claim: *G' is perfect*

Proof: To establish that every proper induced subgraph of G' is perfect we use induction on the number of vertices of G. **Base Case:** Consider the vertex expansion of K_1 into K_2. Both K_1 and K_2 are perfect. Therefore, base case is satisfied.

Inductive step: Let H be any proper induced subgraph of G'. Then H can be isomorphic to any proper induced subgraph of G or can contain x' as one its vertices. If H is isomorphic to any proper induced subgraph of G then H is perfect since G is perfect and every induced subgraph of G is also perfect. If H contains x' then two cases arises. Either H contains both x and x' or H contains only x'.

Case 1: H contains only x'

If H contains only x' then x' can be replaced by x to get a new graph H' since, x' is connected to all neighbours of x. H and H' are isomorphic. H' is perfect since it is a proper induced subgraph of G. Therefore, H is also perfect since it is isomorphic to H'.

Case 2: H contains both x and x'

$H - \{x'\}$ is perfect because it is a proper induced subgraph of G. Therefore, H is a vertex extension of $H - \{x'\}$ and by the inductive hypothesis H is also perfect.

Therefore, any proper induced subgraph H of G' is perfect.

The maximum clique size of G' can increase by atmost 1. Therefore, $\omega(G')$ can be either $\omega(G)$ or $\omega(G) + 1$. We require atmost 1 extra colour to colour G'. Therefore,

$$\chi(G') \leq \chi(G) + 1 \tag{1}$$

Case 1: If $\omega(G') = \omega(G) + 1$

Since, G is perfect $\omega(G) = \chi(G)$. Therefore, (1) can be written as

$$\chi(G') \leq \omega(G) + 1 = \omega(G') \tag{2}$$

Therefore,

$$\chi(G') \leq \omega(G') \tag{3}$$

Case 2: If $\omega(G') = \omega(G)$

x does not lie in any of the maximum cliques of G (If we assume that it lies in any maximum clique K_w then $K_w + \{x'\}$ would form a clique of size $\omega(G) + 1$ as x' is connected to x and all the neighbours of x. Then, $\omega(G')$ is equal to $\omega(G) + 1$ which is a contradiction). Let K_w be any arbitary maximum clique of G. Then the colour class X of x must meet all such K_w because we need $\omega(G)$ colours to colour both G and K_w as G is perfect and each vertex of K_w

will get a different colour so, there will be a vertex in each of K_w such that it has the same colour of x. Let H be $G - (X - \{x\})$ then,

$$\omega(H) = \omega(G) - 1 \tag{4}$$

Because every maximum clique in G must lose a vertex and x cannot be connected to all the vertices of any maximum clique of H (If x was connected then it would be part of a maximum clique of G which is a contradiction). Since, H is a proper induced subgraph of G it is perfect. Therefore, it can coloured with $\omega(H)$ colours. As X is a colour class it is independent. Therefore, $X - \{x\} \cup \{x'\}$ is also independent by virtue of the construction of x'. Therefore, $X - \{x\} \cup \{x'\}$ can be coloured with a single colour. Therefore, $\omega(H)$ colouring of H can be extended into a $\omega(H) + 1$ colouring of G'. Therefore,

$$\chi(G') \leq \omega(H) + 1 \tag{5}$$

Using (4) the expression (5) becomes

$$\chi(G') \leq \omega(G) \tag{6}$$

Since, $\omega(G) = \omega(G')$ the expression (6) becomes

$$\chi(G') \leq \omega(G') \tag{7}$$

Since, both cases satisfy $\chi(G') \leq \omega(G')$ and $\chi(G') \geq \omega(G')$ because, for any graph we need atleast as many colours as the maximum clique size to colour it we can conclude that,

$$\chi(G') = \omega(G') \tag{8}$$

Since every proper induced subgraph of G' is perfect and by using expression (8) we can conclude that G' is perfect. This completes the proof that G' is perfect. □

12.3 Existence of a complete subgraph that meets all maximum independent sets in a perfect graph

Claim: *Any arbitary perfect graph has a complete subgraph that meets all the maximum independent sets.*

Proof: Let G be any arbitary perfect graph. Let κ denote the set of all vertex sets of complete subgraphs of G and A be the set of all maximum independent vertex sets in G. For the sake of contradiction we assume to the contrary that there is no such complete subgraph $\mathrm{K} \in \kappa$. Then for every complete subgraph $\mathrm{K} \in \kappa$ of G we must have some maximum independent set A_K of G so that $\mathrm{K} \cap A_K = \phi$. If any vertex x of G is in a A_K then we count such K's to get $k(x)$, the size of the clique that extends G at vertex x. Let G' be the resulting extension of G. So, vertex x of G may vanish in G' if $k(x) = 0$ but this does not affect perfectness of the extension because, removal of any vertex will result in a proper induced subgraph of a perfect graph and therefore, it is perfect. Therefore, G' can be construted from repeated vertex extension of an induced subgraph of G. Therefore, G' is perfect. Therefore,

$$\chi(G') = \omega(G') \tag{1}$$

Let G_x denote the complete graph extension at a vertex x. By construction of G', every maximal complete subgraph of G' has the form $G'\left[\bigcup_{x\in X} G_x\right]$ for some $K \in \kappa$. So there exists a set $X \in \kappa$ such that

$$\omega(G') = \sum_{x\in X} k(x) \tag{2}$$

This can be written as,

$$\omega(G') = |\{(x, \mathrm{K}) : x \in X, \mathrm{K} \in \kappa, x \in A_{\mathrm{K}}\}| \tag{3}$$

Expression (3) can be rewritten as,

$$\omega(G') = \sum_{\mathrm{K}\in\kappa} |X \cap A_{\mathrm{K}}| \tag{4}$$

Since, X is a complete graph and A_{K} is an independent set $|X \cap A_{\mathrm{K}}| \leq 1$ and $|X \cap A_X| = 0$ by the choice of A_X. Therefore, expression (4) becomes

$$\omega(G') \leq |\kappa| - 1 \tag{5}$$

On the other hand,

$$|G'| = \sum_{x\in V} k(x) \tag{6}$$

This can be written as,

$$|G'| = |\{(x, \mathrm{K}) : x \in V, \mathrm{K} \in \kappa, x \in A_{\mathrm{K}}\}| \tag{7}$$

Expression (7) can be rewritten as,

$$|G'| = \sum_{\mathrm{K} \in \kappa} |A_{\mathrm{K}}| \tag{8}$$

As A_{K} is a maximum independent set of G expression (8) becomes

$$|G'| = |\kappa|.\alpha(G) \tag{9}$$

Since, $\alpha(G') \leq \alpha(G)$ because by construction of G' no new vertices can be added into any independent set of G but some vertices can vanish. Therefore, expression (9) becomes

$$|G'| \geq |\kappa|.\alpha(G') \tag{10}$$

For any graph $\chi(G').\alpha(G') \geq |G'|$ therefore, expression (10) becomes

$$\begin{aligned} \chi(G').\alpha(G') &\geq |\kappa|.\alpha(G') \\ \chi(G') &\geq |\kappa| \\ \chi(G') &> |\kappa| - 1 \end{aligned} \tag{11}$$

Using (5) the expression (11) becomes

$$\chi(G') > \omega(G') \tag{12}$$

Expression (12) is a contradiction to expression (1). Therefore, our assumption that there is no complete subgraph of G that doesn't meet any of the maximum independent sets is false. This completes the proof that any arbitary perfect graph has a complete subgraph that meets all the maximum independent sets. □

12.4 Lovász's Theorem

Claim: *A graph is perfect if and only if its complement is perfect.*

Proof: Let G be any arbitary perfect graph and G' be its complement. We use induction on the number of vertices of G to establish the theorem.

Base Case: If $|G| = 1$

$|G| = 1$ implies that G is K_1. Complement of K_1 is K_1. K_1 is perfect trivially. Therefore, base case is satisfied.

Base Case: If $|G| \geq 2$

Every proper induced subgraph of G' is the complement of a proper induced subgraph of G, and is hence perfect by induction. Since, every perfect graph has a complete subgraph that meets all its maximum independent sets, let K be such a complete subgraph of G. As K is a complete subgraph that meets all the maximum independent sets of G it will be some independent set in G' that meets all of its maximum cliques. Therefore, removing K from G' will decrease the maximum clique size by atleast 1.Therefore,

$$\omega(G' - K) < \omega(G') \tag{1}$$

Since, K is an independent set of G' we can colour it with a single colour and we need atmost one extra colour from $\chi(G' - K)$ colouration of $G' - K$ to colour G'. Therefore,

$$\chi(G') \leq \chi(G' - K) + 1 \tag{2}$$

Using the inductive hypothesis the expression (2) becomes

$$\chi(G') \leq \omega(G' - K) + 1 \tag{3}$$

Using (1) the expression (3) becomes

$$\chi(G') \leq \omega(G') \tag{4}$$

Since, for any graph we need atleast as many colours as the maximum clique size to colour it we can write,

$$\chi(G') \geq \omega(G') \tag{5}$$

Using both (4) and (5) we can write,

$$\chi(G') = \omega(G') \tag{6}$$

Since every proper induced subgraph of G' is perfect and by using expression (6) we can conclude that G' is perfect. This completes the proof for the theorem. □

References

[Lov72] László Lovász. "On the structure of factorizable graphs". In: *Acta Mathematica Hungarica* 23.3-4 (1972), pp. 179–195.

[Lov79] László Lovász. "On the Shannon capacity of a graph". In: *IEEE Transactions on Information Theory* 25.1 (1979), pp. 1–7.

[Wes01] Douglas B. West. *Introduction to Graph Theory*. 2nd ed. Pearson, 2001.

[Chu+02] Maria Chudnovsky et al. “The strong perfect graph conjecture”. In: *Annals of Mathematics* 157.2 (2002), pp. 941–1007.

[Chu+06] Maria Chudnovsky et al. “The strong perfect graph theorem”. In: *Annals of Mathematics* 164.1 (2006), pp. 51–229.

[Die17] Rienhard Diestel. *Graph Theory*. 5th ed. Springer, Springer Nature, 2017.