Advanced graph theory: Mid-Semester Examination : CS60047 Autumn 2024

2:00 pm to 5:00 pm: Marks: 100
Attempt any five out of the questions numbered one through eight, and any four out of the remaining.

September 23, 2024

1. Show that we can arrange the vertices of a connected n-vertex graph G as $x_1, x_2, ..., x_n$ and greedily color them one by one starting from x_1 and ending in x_n, to get a $1 + \Delta(G)$ coloring where $\Delta(G)$ is the maximum vertex degree in G. [5 marks]

   If a connected graph G is a not a regular graph, then can we properly color the vertices of G with at most $\Delta(G)$ colors? Prove your answer. (7 marks)

2. Suppose we sort the vertices of a connected graph G in non-increasing order of their vertex degrees as $x_1, x_2, ..., x_n$, so that $degree(x_1) = \Delta(G)$ and $degree(x_n) = \delta(G)$. Show that we can properly color the vertices of G using at most $1 + max_{i=1}^{n} min(i-1, degree(x_i))$ colors. (4 marks)

   Also, show that $\chi(G) \leq n - \alpha(G) + 1$, where n is the number of vertices of G. Here $\chi(G)$ is the (vertex) chromatic number of G and $\alpha(G)$ is the size of the maximum independent set of G. (8 marks)

3. Show that the vertex connectivity of a graph is at most the edge connectivity. (6 marks)

   Determine the maximum and minimum values of vertex and edge connectivity for a split graph in terms of the chromatic number of the graph. (6 marks)

4. Show that a *comparability graph* is a perfect graph. A graph is said to be a *comparability graph* if there is a partial order on its vertices and there is an edge between a pair of vertices if and only if that pair of vertices is related in that partial order. You may use Mirsky's theorem which essentially states that the longest path of related vertices in the partial order is equal to the size of the minimum partition into sets of vertices where each set in the partition has mutually unrelated vertices. (6 marks)

Now assuming comparability graphs are perfect and applying the (weak) perfect graph theorem (PGT), show that the maximum cardinality set of incomparable vertices in a partial order is the same as the size of the minimum cardinality partition of the vertex set where each set in the partition is a set of mutually comparable vertices in the partial order. (6 marks)

5. Using Hall's theorem show that a k-regular bipartite graph has a perfect matching. (6 marks)

   Verify whether Tutte's necessary and sufficient condition holds for (i) even cycles, (ii) odd cycles, (iii) complete bipartite graphs. (6 marks)

6. Suppose G' is an undirected and connected simple graph with an even number of vertices but has no perfect matching. Assume that G' is such that adding any edge e to it causes $G' + e$ to have a perfect matching. Let S be a subset of vertices so that the induced subgraph $G' - S$ has more than $|S|$ odd components. Determine whether any component of the induced subgraph $G' - S$ is a complete subgraph. Determine whether S induces a complete subgraph in G'. Determine which vertices of S have edges in G' with how many vertices in the components of $G' - S$. (4+4+4 marks)

7. Use Hall's theorem to show that the size of the smallest vertex cover in a bipartite graph is equal to the size of the maximum matching. (12 marks)

8. Given a graph G, show that there is a *spanning bipartite subgraph* H of G such that the vertex degrees in H are at least half their respective vertex degrees in G. (12 marks)

   [Hint:]

9. Given a tree T show that its complement $\bar{T}$ is a perfect graph. (10 marks) [Hint: This follows readily from the perfect graph theorem. However, you must establish this by using strong induction, the Konig-Egervary theorem and Gallai's theorem.]

10. Assume that two chordal graphs can be combined and united by pasting them along their common complete subgraph to result in another chordal graph. Also, complete subgraphs are chordal. Assume further that any chordal graph can be thus constructed by combining two smaller chordal graphs by pasting them along their common complete graph, starting with complete subgraphs. Based on these assumptions, whereby any chordal graph can be constructed as mentioned, show that chordal graphs are perfect. (10 marks)

    [Hints: Since any chordal graph G can be constructed by pasting some two chordal graphs G_1 and G_2 along their common complete subgraph

S, and that by the induction hypothesis, we may assume that G_1 and G_2 are perfect, all we need to do is show that G is perfect by proving that for every induced subgraph H of G, $\chi(H) \le \omega(H)$. Note that induced subgraphs of G may overlap with G_1 and G_2, whereas all induced subgraphs of each of G_1 or G_2 are perfect, because G_1 and G_2 are perfect by the induction hypothesis.]

11. Show that a 3-regular graph with at most two cut edges has a perfect matching. (10 marks)

    [Hint: For the sake of contradiction assume that there is no perfect matching and S is the bad set. Find whether the parities of the set S and $o(G - S)$ are same or different and write the violated Tutte's condition. Now plug in the requirements that there are at most two cut edges to derive a contradiction.]

12. We wish to show that intersections of subtrees of a tree obey the Helly property, whereby the set $\mathcal{S} = \{T | T \in \mathcal{T}\}$ of pairwise intersecting subtrees of a tree $\mathcal{T}$ also has a non-empty intersection $\cap_{T \in \mathcal{S}} T$. Complete the following argument for establishing this Helly property for intersecting subtrees of a tree.
    (10 marks)

    [Hints: Suppose we use induction on the number k of subtrees of an n-vertex tree $\mathcal{T}$. Conisder k subtrees $T_1, T_2, ..., T_k$ of $\mathcal{T}$ which intersect pairwise. For the sake of contradiction we assume that they do not have a common intersection. However, by the induction hypothesis $T_1, T_2, ..., T_{k-1}$ intersect in say a subtree T_0. As T_k misses T_0 let us find a connecting path P from T_0 to T_k with a vertex $x \in P \cap T_k$ and a vertex y adjacent to x on P closer to T_0. Now $\mathcal{T} - xy$ has connected components where the edge xy separates T_0 from T_k.]

13. Show that a graph G is perfect if and only if it has the property that every induced subgraph H contains an independent set $A \subseteq V(H)$ such that $\omega(H - A) < \omega(H)$. (10 marks) [Hint: Use induction.]

14. Show that $\chi(G) + \chi(G') \le n+1$ for an n-vertex undirected graph where G' is the complement graph of G. (10 marks)