Coverage

November 1, 2024

Date	Covered
31.07.24	**1**. Prove: "A graph is bipartite $\Leftrightarrow \exists$ no odd cycle". **2**. Edge Connectivity $[\lambda(G) \leq \delta(G)]$. **3**. Vertex Connectivity $[\kappa(G) \leq \lambda(G)]$. **4**. Show that the existence of $n-k$ distinct paths of length k in a tree T with diameter $2k-3$. **5**. Complement graph. **6**. Prove: In regular graph $\lvert E\rvert = \frac{\Delta(G)}{2} \cdot \lvert V\rvert$, where $\alpha \leq \frac{\lvert V\rvert}{2}$. **7**. [Bounding triangles in a graph]: Show that the # triangles in any simple graph of n vertices and m edges is at least $\frac{4m}{3n} \cdot (m - \frac{n^2}{4})$.
01.08.24	**1**. Def(4.1.7) [1]: Disconnecting Set of edges, Edge cut. **2**. Remark(4.1.8) **3**. Th(4.1.9) [1](Whitney[1932]): If G is a simple graph, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$. [1]. **4**. Ex(4.1.10) [1] **5**. Th(4.1.11) [1]: If G is a 3-regular graph, then $\kappa(G) = \kappa'(G)$. **6**. The proof of the Konig-Egervary Theorem using Hall's theorem.
02.08.24	**1**. Proof of Hall's Theorem using the Konig-Egervary theorem. **2**. Revision of the proofs of Mantel's theorem. **3**. Tutorial 1 and discussions on the tutorial.

07.08.24	**1**. Lm(1.2.25) [1]: If vertex of graph G has degree at least 2, then G contains a cycle. **2**. Prop (1.2.28) [1]: If G is a simple graph in which every vertex has degree at least k, then G contains a path of length k. If $k \geq 2$, then G also contains a cycle of length at least $k+1$. **3**. Cor(1.3.4) [1]: In a graph G, the average vertex degree is $\frac{2\|E\|}{n}$, and hence $\delta(G) \leq \frac{2\|E\|}{n} \leq \Delta(G)$. **4**. Cor(1.3.6) [1]: A $k-$regular graph with n vertices has $\frac{nk}{2}$ edges. **5**. Prop(1.3.9) [1]: If $k > 0$, then a $k-$regular bipartite graph has the same number of vertices in each partite set, and the graph also has a perfect matching. **6**. Prop(1.3.15) [1]: If G is a simple n-vertex graph with $\delta(G) \geq (n-1)/2$, then G is connected.
08.08.24	**1**. Proof of Konig-Egervary theorem using alternating paths. **2**. Th(2.1.1) [2] (Konig[1931]): Max$\|M\|$ =Min$\|VC\|$ i.e. $\alpha' = \beta$.
09.08.24	**1**. Berge's Theorem. **2**. Overview of Tutte's Theorem. **3**. Proof of the necessity of Tutte's condition. **4**. Outline of the main ideas in the sufficiency proof.
14.08.24	**1**. The maximal graph idea for proving Tutte's theorem. **2**. Properties of the universal set of vertices of a maximally saturated graph with no perfect matching.
16.08.24	**1**. Sketch of the proof of Hall's theorem by induction.
21.08.24	**1**. (Cont...) Properties of the universal set of vertices of a maximally saturated graph with no perfect matching, P-84,85

22.08.24	**1**. Overview of the proof of Gallai's theorem: $\beta' = \alpha'$. **2**. The maximal graph idea for proving Tutte's Theorem. **3**. Completion of the details of the equivalence classes property of components of $G \setminus V_1$, by using Berge's theorem.
28.08.24	**1**. Proof of Gallai's theorem using stars and matchings. **2**. Proof of Hall's Theorem by induction. * **Exercise**: Tutte's Theorem ⇒ Hall's Theorem. * **Exercise**: Tutte's Theorem by Induction.
29.08.24	**1**. Th(5.1.1) [2]: Every planar graph is 4-colorable. **2**. Prop(5.1.2) [2]:Every planar graph is 5-colorable. **3**. Th(5.1.1) [2](Grotzsch[1959]): Every planar graph not containing a triangle is 3-colorable. **4**. Prop(5.2.1) [2]: Every graph G with m edges satisfies $\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$. **5**. Th(5.1.19) [1] or Prop(5.2.2) [2](Szekeres-Wilf[1968]): Every graph G satisfies $\chi(G) \leq col(G) = max\{\delta(H) \mid H \subseteq G\} + 1$. **6**. Th(5.2.4) [2](Brooks[1941]): Let G be a connected graph. If G is neither complete nor an odd cycle then $\chi(G) \leq \Delta(G)$. **7**. Perfect Graphs: **8**. Induced subgraphs of perfect graphs are perfect; $\chi(H) = \omega(H)$. **9**. PG Conj(Berge 1966): A graph G is perfect iff neither G nor $\bar{G}$ contains an odd cycle of length at least 5 as an induced subgraph. **10**. Generating perfect graphs by extension of a vertex.

04.09.24	**1**. (Cont...) Generating perfect graphs by extension of a vertex. **2**. Characterizing perfect graphs using overlaps of all maximum cliques with an independent set.
05.09.24	**1**. Proof that bipartite graphs and their complements are perfect graphs. **2**. Generating a perfect graph by the extension at vertices with perfect graphs.
06.09.24	**1**. (Contd.) Generating a perfect graph by the extension at vertices with perfect graphs. * **Exercise**: Chordal graphs and their complements are perfect. * **Exercise:** From the slides of "Proving PGT as in [Die17; GGL95]", (I) Why is $\alpha'(G) \leq \alpha(G)$? (II) Why $\chi(G') \geq \kappa$? (III) Prove: X being a complete subgraph cannot contribute more than one vertex to any A_K for $K \in \kappa$, whereas $X \cup A_X = \phi$, hence $\omega(G') \leq \|\kappa\| - 1$.
11.09.24	**1**. Prop(5.5.1) [2]: A graph is chordal if and only if it can be constructed recursively by pasting along complete subgraphs, starting from complete graphs. **2**. Prop(5.5.2) [2]: Every chordal graph is perfect. **3**. Th(5.5.4) (Lova´sz [1972]): A graph is perfect iff its complement is perfect. **4**. Lm(5.5.5) Any graph obtained from a perfect graph by expanding a vertex is again perfect. **5**. Th(5.5.6) [2]: A graph G is perfect iff $\|H\| \leq \alpha(H).\omega(H)$ for every induced subgraph H of G. **6**. Induced subgraphs of perfect graphs are perfect.

12.09.24	**1**. Chromatic index or Edge coloring (χ'): **2**. The chromatic index of a bipartite graph G is always equal to the maximum degree of its vertices; $\chi'(G) = \Delta(G)$. **3**. If $L(G)$ is the line graph of G, then $\chi'(G) = \chi(L(G))$. **4**. $\chi'(G) \geq \Delta(G)$. **5**. $\chi'(G) \geq \left\lceil \frac{e(G)}{\beta(G)} \right\rceil$. **6**. The number of edges should increase by at least $\chi(G')$ to at least $\chi(G') + \binom{\chi(G')}{2} = \binom{\chi(G')+1}{2}$ in the new graph. **7**. In any induced subgraph H of a perfect graph G, we have $\chi(H) = \omega(H) \geq \frac{n(H)}{\alpha(H)}$. **8**. In a proper greedy coloring $\chi(G)$ does not exceed $\Delta(G)+1$.
13.09.24	**1**. Mentioned about Erdos-Stone theorem. **2**. Overview of (I) Turan's problem and (II) the problem of K. Zarenkiewicz with an application from geometry.
26.09.24	**1**. Introducing Turan's extremal graph, revision. Construction and counting edges by induction. **2**. Sketch of the proof of Hall's theorem using Tutte's theorem.
27.09.24	**1**. Sufficient conditions for ascertaining Hamiltonian circuits in graphs. **2**. Allusion to problems about graphs avoiding cycles of size r. History of the genesis of Extremal graph theory under the tulilage of Konig. Review of the extremality of Turan graphs and the uniqueness of Turan graphs. Allusion to the exclusion of complete bipartite graphs as in the problem of K. Zarenkiewicz.

03.10.24	**1.** Th(9.3) [3] (Turan): Let G be a K_r-free graph with n vertices. Then $\|E(G)\| \leq \|E(T_{r-1}(n)\|$, with quality iff G is isomorphic to $T_{r-1}(n)$. **2.** Th(9.4) [3] (Erdos): Given any K_r-free graph G with n vertices, one can always constract an $(r-1)$-partite graph H on the same vertex set such that $d_H(x) \geq d_G(x)$ $\forall x \in V(G) = V(H)$. * **Exercise**: For a given graph G with n vertices if you have K_4^t, is it possible to get K_5^t by adding one edge?
04.10.24	**1.** Th(9.10) [3] (Erdos and Stone): Let $r \geq 2$, t be fixed natural numbers, $\epsilon > 0$. Then $\exists$ an integer $n_0 = n_0(r, t, \epsilon)$ such that any graph G with $n \geq n_0$ vertices and at least $\frac{n^2}{2} \cdot (1 - \frac{1}{r-1} + \epsilon)$ edges has a complete r-partite subgraph $K_{t,t,\ldots t}$ whose classes are of size t. **2.** Lm(9.11) [3]: $\exists$ $n_1 = n_1(r, t, \epsilon)$ such that if G has $n \geq n_1$ vertices and $d(x) \geq n(1 - \frac{1}{r-1} + \epsilon)$ $\forall x \in V(G)$, then $G \supseteq K_{t,t,\ldots t(r-times)}$. **3.** Cor(9.12) [3]: Given a non-empty graph H with chromatic number $\chi(H)$, $ex(n, H) = \frac{n^2}{2}(1 - \frac{1}{\chi(H)-1}) + o(n^2)$. **4.** Th(9.1) [3] (Dirac): Let G be a connected graph with $n \geq 3$ vertices such that $d(x) + d(y) \geq 1$ for every pair of non-adjacent vertices $x, y \in V(G)$, (I) If $r = n$, then g has Hamiltonian Cycle; (II) If $r < n$, then $G \supseteq P_r$ and $G \supseteq C_{\lceil (r-1)/2 \rceil}$. **5.** Th(7.1.1) [2] (Turan[1941]): Foe all integer r, n with $r > 1$, every graph $G \not\supseteq K_r$ with n vertices and $ex(n, K_r)$ edeges is a $T_{r-1}(n)$. **6.** Th(7.1.2) [2] (Erdos and Stone[1946]): For all integer $r \geq 2$ and $s \geq 1$, and every $\epsilon > 0$, $\exists$ an integer n_0 such that every graph with $n \geq n_0$ vertices and at least $t_{r-1}(n) + \epsilon n^2$ edeges contains K_r^s as a subgraph. **7.** Cor(7.1.3) [2]: For every graph H with at least one edge, $\lim_{n\to\infty} ex(n, H) \binom{n}{2}^{-1} = \frac{\chi(H) - 2}{\chi(H) - 1}$. * **Exercise**: Is every subgraph of a graph G satisfies Tutte's theorem, where G is a perfect graph contains perfect matching? * **Exercise**:Why perfect graph does not have perfect matching? * **Exercise**: Prove by Induction: (I) Hall's theorem, (II) Konig-Egervary theorem.

16.10.24	**1**. Hypergraph: Vapnik-Chervonenkis dimension (VC-dimension) **2**. Additional ref: [4]
17.10.24	**1**. Discussion on mid-sem questions.
18.10.24	**1**. Tutorial 4, HW 3, Ore's theorem, Proof of Erdos-Stone theorem, and initial discussion on the outline for the proof of Lovasz PGT version based on sizes of maximum independent sets and maximum cliques of all induced subgraphs.
23.10.24	**1**. Planarity: **2**. <u>Th(6.2)</u> [5] (Kuratowski): A graph is planar iff it does not have any subgraph homeomorphic to K_5 or $K_{3,3}$. **3**. <u>Lm(6.2)</u> [5]: A non-planar connected graph G with minimum number of edges that contains no subdivision of K_5 or $K_{3,3}$ is simple and 3-connected. **4**. See also the crucial supporting Proposition 5.26 in [5], attributed to Thomassen (1981,1985), and used in proving the sufficiency condition for planarity by Kuratowski in Theorem 6.2 of [5]. **5**. See Section 6.2 of [1], Lemmas 6.2.7, 6.2.6, 6.2.5 and 6.2.4, in that order for a detailed top-down presentation of the main result about considering only 3-connected graphs, as in Lemma 6.2 of [5]. Definition 6.2.3 for "Kuratowski subgraphs", "minimal non-planar graphs" in [1], and Definition 5.2.19 in [1] will be useful.
30.10.24	**1**. Discussions on term-paper topics and contents.
01.10.24	**1**. Tutorial 5 discussions on the Chapter 6 planarity characterization of graphs as in [1] based Kuratowski's graphs. **2**. Every "minimal" non-planar graph is 2-connected. **3**. Let G be a *"minimal"* non-planar graph with with all edges having degrees at least three. Then show that G is 3-connected. By minimality we mean that each proper subgraph of G is planar. Also, G must be isomorphic to a K_5 or a $K_{3,3}$. **4**. Show that if G is a graph with the fewest number of edges among all non-planar graphs without Kuratowski subgraphs, then G is 3-connected (Theorem 6.2.7 in West's book [1]). **5**. Uisng item 3 above, show that a graph G that is non-planar but 3-connected must have a Kuratowski subgraph.

Table 1: Summary of Lectures.

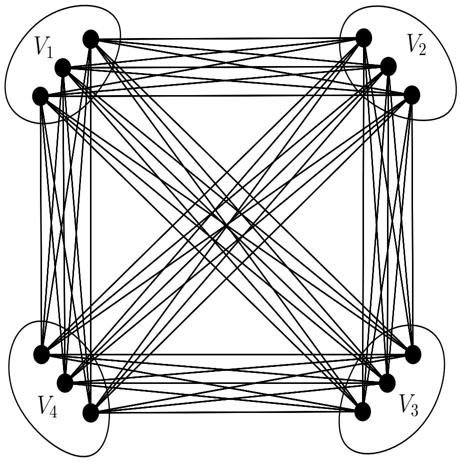

Figure 1: $T_4^3(12)$, the Turan graph of 12 vertices, 4-partite, with three vertices in each partite and thus also the K_4^3. This graph has multiple K_4 but is just one edge deficient from possessing a K_5.

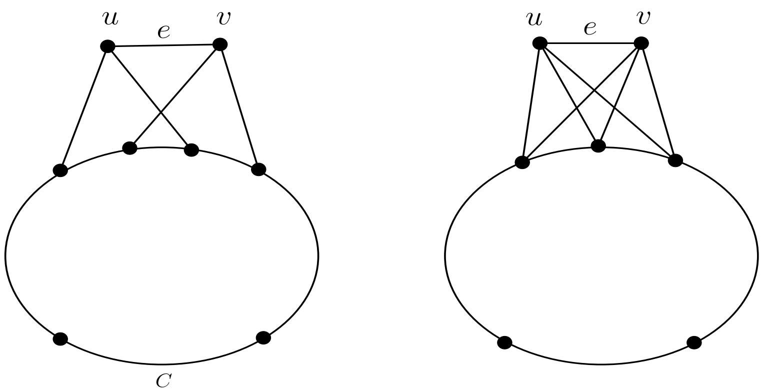

Figure 2: The non-planar cases of $K_{3,3}$ and K_5 respectively, appearing as illustrated in Figure 6.7(a) [5].

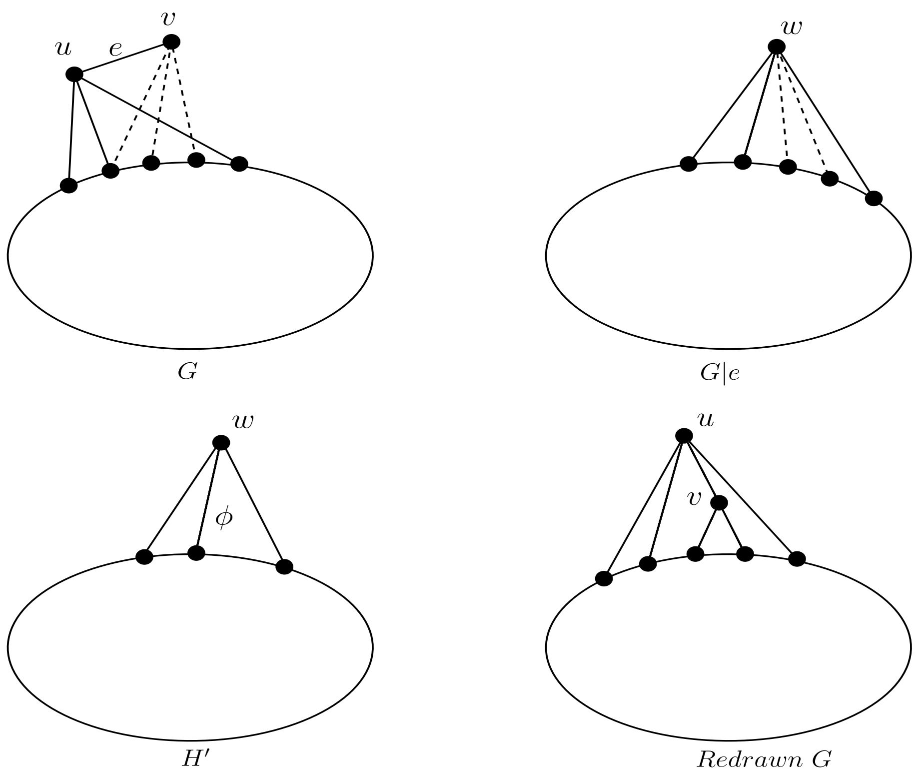

Figure 3: The transformations not violating planarity in the planar case, as illustrated in Figure 6.7(b) [5].

0.1 Class of Wednesday, August 14, 2024

Based on the discussions in the class of August 14, 2024, try the following exercises.

1. Draw bipartite graphs $G(A \cup B, E)$ with no perfect matching, or a maximum matching that does not match all the vertices in A, given that $|A| < |B|$.
2. Explain the dense structure of maximally perfect-matching-free graphs with respect to bad sets vaiolating Tutte's condition.
3. Give examples of graphs having perfect matchings and how all subsets of vertices satisfy the Tutte condition.

0.2 Class of August 16, 2024

Applying Hall's theorem to show that there is a 2-factor in a graph that is $2k$-regular.

Sketch of the proof of Hall's theorem by induction.

0.3 Class of August 21, 2024

Detailed proof of Tutte's theorem: Use is made of the set V_1 of the edge-maximal simple, even undirected graph G' with no perfect matching whereas $G' + e$ has a perfect matching for any edge e not in G'. This set V_1 is the collection of all vertices with degree $n(G') - 1$ in G'. We show that V_1 is a bad set for G'.

0.4 Class of August 22, 2024

Sketch of the proof of Galai's theorem, $\alpha'(G) = \beta'(G)$.

Completion of the details of the equivalence classes property of components of $G \setminus V_1$, by using Berge's theorem.

0.5 Class of August 28, wednesday, 2024

Proof of Galai's theorem, $\alpha'(G) = \beta'(G)$ using stars and matchings.

0.6 Class of August 29, thursday, 2024

Perfect graphs: introduction, motivation and constructions.

0.7 Test 1: August 30

0.8 Class of September 04, 2024

Constructing perfect graphs by extensions of vertices.

Characterizing perfect graphs by the existence of an independent set in every induced subgraph that meets all maximum cliques in that induced subgraph.

0.9 Class of September 05, 2024

Complements of bipartite graphs are perfect; proof using Galai's theorem.
Extensions of perfect graphs at vertices by arbitrary perfect graphs generate perfect graphs.
Exercise: Prove that chordal graphs are perfect using the property that they are triangulated.
Also show the existence of *simplicial* vertices and establish their properties.

0.10 Class of September 06, 2024

0.11 Class of September 11, 2024

Building chordal graphs from cliques by pasting two chordal graphs on a common complete subgraph.
Using the above structural construction of chordal graphs to show that they are perfect.
These are from [2].
We also considered the proof of PGT again, as in [2].

0.12 Class of Friday, October 04, 2024

Revision of the proof of Hall's theorem using the Konig-Egervary theorem.
The theorems of Turan and Erdos-Stone, and their proof sketches.

Bibliography

[1] D. B. West *et al.*, *Introduction to graph theory.* Prentice hall Upper Saddle River, 2001, vol. 2.

[2] R. Diestel, *Graph theory.* Springer (print edition); Reinhard Diestel (eBooks), 2024.

[3] J. Pach and P. K. Agarwal, *Combinatorial geometry.* John Wiley & Sons, 2011.

[4] O. Filtser, E. Krohn, B. J. Nilsson, C. Rieck, and C. Schmidt, "Guarding polyominoes under k-hop visibility," in *Latin American Symposium on Theoretical Informatics.* Springer, 2024, pp. 288–302.

[5] K. R. Parthasarathy, *Basic Graph Theory.* Tata McGraw-Hill Publishing Company Limited, 1994.