CS60047 Advanced Graph Theory	2023-09-14
Lecture (12)	
Lecturer: Prof. S. P. Pal	*Scribe: (Bhagoti)*

12 Topic Overview

In this Lecture, we delve into graph coloring. The Greedy Algorithm for vertex coloring is introduced as a practical method for coloring graphs. including Vizing's theorem, which sets limits on edge coloring. Planar graphs, known for their distinctive properties, can be effectively colored with just three colors when devoid of triangles, as per the Five Color Theorem. Explore interval graphs and their relation to chordal graphs, and how the complement of an interval graph transforms into a comparability graph.

Every planar graph not containing any triangle is 3 colorable. (girth >4, 5 ....) and If we increase the girth the chromatic number will become large (if the graph is not planar)

Theorem 12.0.1. *Show that Every planar graph is 5-colorable. [3]*

Proof. As given in the book Narsingh Deo G is a planar graph with n vertices. and we have to proof that G is 5 colorble. for this, we will use induction. we know that if a graph is planar then a vertex has a degree 5 or less. because−

$$Deg(G) = \frac{2m}{n} \leq \frac{2(3n-6)}{n} < 6$$

induction base case if graph G has less than or equal to 5 vertices than we can color this graph by 5 colors. so this theorem holds for the base case. Now **induction hypothesis** assumes that it is every planar graph G with vertices $n-1$ can be properly colored by 5 colors. then we will proof that planar graph G with n vertices can be colored by 5 colors.

As we discussed above for a planar graph G a vertex has a degree less than or equal to 5 assume that this is vertex v.
let G' be a graph of $n-1$ vertices obtained from G deleting by vertex v. and this requires no more than 5 colors(induction hypothesis) for proper coloring.Now we add a vertex to graph G'. if the degree of this vertex is less than 5 $(1, 2, 3, 4)$ then there is no difficulty in assigning color to this vertex, and in this case, there will not use more than 5 colors for proper coloring of planar graph G.
the case in which vertex has degree 5 and all the 5 colors have been used coloring to adjacent vertices of v.Suppose that there is a path in G' between vertices a and c colored alternately with colors 1 and 3. Then a similar path between b and d, colored alternately with colors 2 and 4, cannot exist; otherwise, these two paths will intersect and cause G to be nonplanar. If there is no path between b and d colored alternately with colors 2 and 4, starting from vertex b we can interchange colors 2 and 4 of all vertices connected to b through vertices of alternating colors 2 and 4. This interchange will paint vertex b

Figure 1: Reassigning of colors

with color 4 and yet keep G' properly colored. Since vertex d is still with color 4, we have color 2 left over with which to paint vertex v.

$\square$

12.1 Coloring vertices

Theorem: every graph G with m edges satisfies

$$\chi(G) \leq \frac{1}{2} + \sqrt{2m + 1/4}$$

Proof. Let c be a vertex coloring of G with $k = \chi(G)$ colors. Then G has at least one edge between any two color classes: if not, we could have used the same color for both classes. Thus,

$$m \geq \frac{1}{2}k(k-1)$$

.

$$= 2m \geq k(k-1)$$
$$= 2m \geq k^2 - k$$

Add $\frac{1}{4}$ to both sides.

$$= 2m + \frac{1}{4} \geq (k - \frac{1}{2})^2$$
$$= \sqrt{2m + \frac{1}{4}} \geq (k - \frac{1}{2})$$
$$= k \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$$

as $k = \chi(G)$ thus we proved that

$$\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$$

Theorem: Every graph G satisfies

$$\chi(G) \leq \mathrm{col}(G) = \max\{\delta(H) \mid H \subseteq G\} + 1$$

12.1.1 Greedy Algorithm:[2]

Color a graph G with not too many colors is the following greedy algorithm-
Starting from a fixed vertex enumeration $v_1, v_2, ...v_n$ of G, we consider the vertices in turn and color each v_i with the first available color - e.g., with the smallest positive integer not already used to color any neighbor of v_i among $v_1, ...v_(i-1)$. In this way, we never use more than $\Delta(G)+1$ colors. even for unfavorable choices of the enumeration $v_1, ...v_n$. If G is complete or an odd cycle, then this is even the best possible.

this upper bound of $\Delta + 1$ is rather generous, even for greedy colorings. Indeed, when we come to color the vertex v_i in the greedy algorithm, we only need a supply of $d_G[v1, ..., vi](vi)+1$ rather than $d_G(vi)+1$ colors to proceed; recall that, at this stage, the algorithm ignores any neighbors vj of vi with $j{>}i$. Hence in most graphs, there will be scope for an improvement of the $\Delta + 1$ bound by choosing a particularly suitable vertex ordering to start with: one that picks vertices of large degree early (when most neighbours are ignored) and vertices of small degree last. Locally, the number $d_G[v1, ..., vi](vi) + 1$ of colors required will be smallest if vi has a minimum degree in $G[v1, ..., vi]$. But this is easily achieved: we just choose v_n first, with $d(v_n) = \delta(G)$, then choose as $v_n - 1$ a vertex of minimum degree in $G - v_n$, and so on.

12.1.2 Coloring number $(col(G))$

The least number k such that G has a vertex enumeration in which each vertex is proceeded by fewer than k of its neighbors is called the coloring number $col(G)$ of G.

$$col(G) \leq max_(H \subseteq G)\delta(H) + 1$$

12.1.3 Every k-chromatic graph has a k chromatic subgraph of minimum degree at least $k - 1$

Proof: Given G with $\chi(G) = k$
Let, $H \subseteq G$ be minimal with $\chi(H) = k$.
if H had a vertex v of degree $d_H(v) \leq k-2$, we could extend a $(k-1)$ coloring of $H - v$ to one of H, contradicting the choice of H.

12.2 Coloring edges

Theorem 12.2.1. *Every bipartite graph G satisfies $\chi'(G) = \Delta(G)$[konig][2]*

Proof. As given in Diestel We apply induction on $|G|$. For $|G| = 0$ the assertion holds. Now assume that $|G| \geq 1$, and that the assertion holds for graphs with fewer edges.

induction hypothesis: Let $\Delta := \Delta(G)$, pick an edge $xy \in G$, and choose a δ edge-coloring of $G - xy$. let us the edge colored α as α-edges. In $G - xy$, each x and y is incident with at most $\Delta - 1$ edges.Hence there are $\alpha, \beta \in 1, ...\Delta$ such that x is not incident with an α - edge and y is not incident with β - edge.
if $\alpha = \beta$ we can color the edge xy with the color and are done, so we assume that $\alpha \neq \beta$ and x are incident with a β - edge. Let us extend this edge to a maximal walk W from x

whose edges are colored β and α alternately. Since no such walk contains a vertex twice (why not?), W exists and is a path. Moreover, W does not contain y: if it did, it would end in y on an α-edge (by the choice of β) and thus have an even length, so $W + xy$ would be an odd cycle in G. We now recolor all the edges on W, swapping α with β. By the choice of α and the maximality of W, adjacent edges of $G - xy$ are still colored differently. We have thus found a Δ-edge-colouring of $G - xy$ in which neither x nor y is incident with a β-edge. Coloring xy with β, we extend this coloring to a Δ-edge-colouring of G.
we color the edges of a graph so two edges sharing the same vertex don't have the same color it is called edge coloring($\chi'(G)$).every graph G satisfies $\chi'(G) \geq \Delta(G)$ For bipartite graphs $\chi'(G) = \Delta(G)$ □

12.2.1 Show that every graph G has a vertex ordering for which the greedy algorithms $\chi(G)$

colors.
Answer: Consider an optimal coloring f. Number the vertices of G as $v1, ..., vn$ as follows:
start with the vertices of color 1 in f, then those of color 2, and so on.
By induction on i, we prove that the greedy algorithm assigns vi a color at most $f(vi)$. Certainly, $v1$ gets color 1.
For $i > 1$, the induction hypothesis says that vj has received color at most $f(vj)$, for every $j < i$.
Furthermore, the only vertices vj with $f(vj) = f(vi)$ are those in the same color class with vi in the optimal coloring, which are not adjacent to vi.
Hence the colors used on earlier neighbors of v_i are in the set $1, ..., f(v_i) - 1$, and the algorithm assigns color at most $f(v_i)$ to v_i.

Question . A graph G is called an interval graph if there exists a set $\{I_v \mid v \in V(G)\}$ of intervals such that $I_u \cap I_v \neq \emptyset$ if $uv \in E(G)$ [1].
(i) show that every interval graph is a chordal graph
(ii)show that the complement of any interval graph is a comparability graph.
Answer: Interval Graph: An interval graph is an undirected graph formed from a set of intervals in real-time.
For the chordal graph following properties -
a) Graph G has a simplicial elimination ordering.
b) Graph G has no chordless cycle.
c) every minimal vertex separator of G induces a clique.
Consider an interval representation of G, with each v represented by the interval $I(v) = [a(v), b(v)]$. Let v be the vertex with the largest left endpoint $a(v)$.
The intervals for all neighbors of v contain $a(v)$, so in the intersection graph the neighbors of v form a clique. Hence v is simplicial. If we delete v and proceed with the remainder of the representation, which is an interval representation of $G - v$, we inductively complete a perfect elimination ordering.
Alternatively, let C be a cycle in G. Let u be the vertex in C whose right endpoint is smallest, and let v be the vertex whose left endpoint is largest. If u, v are nonadjacent, then the intervals for the two u, v-paths in C must cover $[b(u), a(v)]$. Hence the intersec-

tion graph has a chord of C between them. We conclude that an interval graph has no chordless cycle.
If $uv \in E(G)$, then $I(u)$ and $I(v)$ are disjoint. Orient the edge uv toward the vertex whose interval is to the left. This yields a transitive orientation of G; if $I(u)$ is to the left of $I(v)$, and $I(v)$ is to the left of $I(w)$, then $I(u)$ is to the left of $I(w)$.
So Interval graphs are chordal graphs.

(ii).
Every interval graph is a chordal graph and is the complement of a comparability graph. If it is not a chordal graph, then it has a chordless cycle. A chordless cycle has no interval representation because the two paths along the cycle between the vertices corresponding to the leftmost and rightmost intervals among these vertices must occupy all the space between them on the line, which produces chords between the two paths when the intersections are taken. Hence the full graph has no interval representation.
Given an interval representation of a graph G, orienting G by $x \rightarrow y$, if the interval for x is complete to the right of the interval for y, expresses G as a comparability graph.

What is a comparability graph?
Defination :A comparability graph is a type of graph where vertices represent objects or elements, and edges connect pairs of objects that can be compared or ordered with respect to some criteria or relationship. In other words, it's a graph that shows which objects in a collection can be compared in a meaningful way.

12.3 Coloring:

12.3.1 Line Graph:

it is an intersection of edges of some other graph.
So we can say that edge coloring chromatic number of graph ($\chi'(G)$) is the same as the coloring chromatic number ($\chi(L(G))$) of line graph ($L(G)$) of graph (G).
A chromatic number of the line graph is well-behaved rather than an arbitrary graph.
A trivial lower bound for $\chi'(G)$ is $\Delta(G)$, with equality attained for bipartite graphs.
Another lower bound is $[\frac{e(G)}{\beta(G)}]$

we can say that graph G requires to $\binom{\chi(G)}{2}$
when we construct a graph G and add a new vertex and add edges incident to the new vertex we will call this graph G'.
so there will be steps of vertex inclusion when their chromatic number $\chi(G')$ is increased. The new vertex added is already adjacent vertices of as many $\chi(G')$ different colors in the proper way of coloring of G'
So the chromatic number is increased by one and the edges of the graph are increased by $\chi(G')$.
1) So we can say that the graph should have edges $\binom{\chi(G')+1}{2}$
2) and we know that edges are increased by the $\chi(G')$ so now edges

$$= \binom{\chi(G')}{2} + \chi(G')$$

$$= \frac{(\chi(G') * (\chi(G') - 1))}{2} + \chi(G')$$
$$= \binom{\chi(G') + 1}{2}$$

which is we said that in the above sequence 1

We need at least as many as $\frac{n(G)}{\alpha(G)}$ colors
So in induced subgraph H of graph of G

$$\chi(G) = \omega(G) \leq \frac{n(H)}{\alpha(H)}$$

So it is that $\chi(G)$ will not exceed $\delta(G) + 1$ in a greedy proper coloring.

References

[1] Douglas B. West. *Introduction to Graph Theory*. 2nd ed. Pearson, 2001.

[2] Reinhard Diestel. *Graph Theory*. 5th ed. Springer, Springer Nature, 2017.

[3] Narsingh Deo: Graph Theory with Applications to Engineering & Computer Sciences, Prentice Hall, 1974