CS60021: Scalable Data Mining

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In this Lecture:

- K – means clustering and applications
- Lloyd's algorithm, EM and Limitations.
- K – means ++
- Scalable k – means ++

K – means clustering and applications

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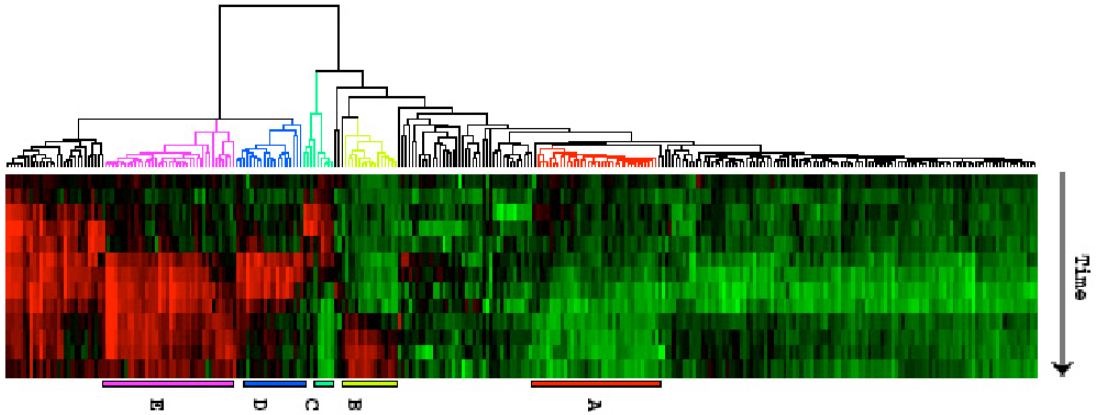
Clustering

- Unsupervised learning
  - When your data doesn't have labels

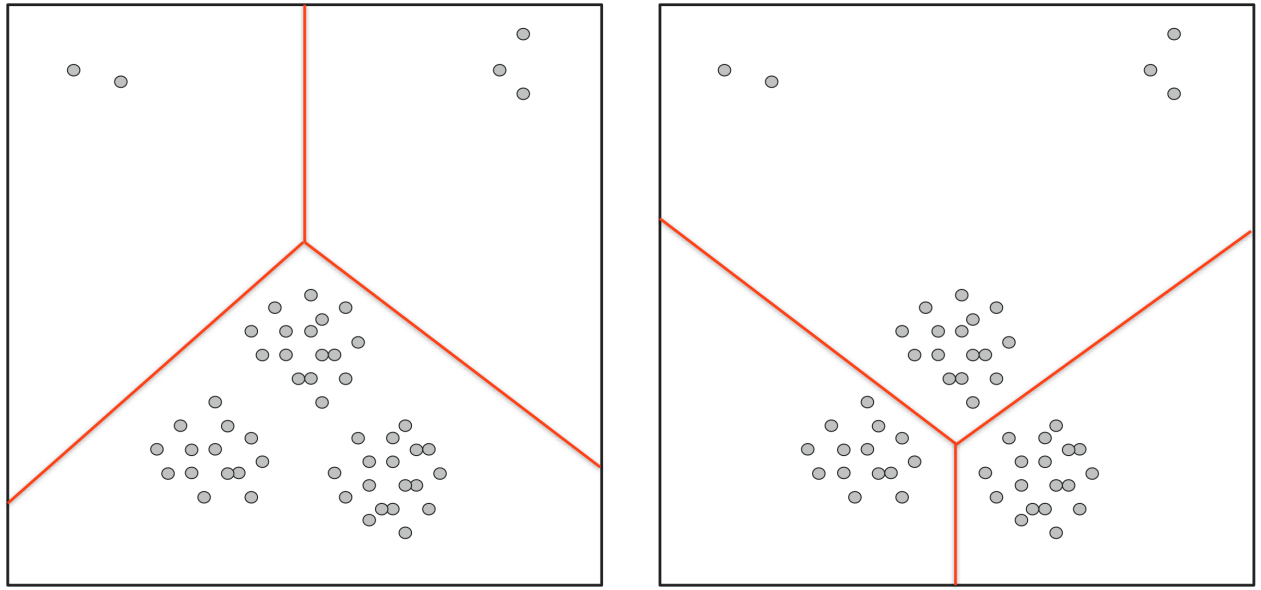
- Useful for
  - Detecting patterns e.g. in image data, customer shopping results, anomalies…
  - For optimizing, e.g. distributing data across various machines, cleaning up search results, facility allocation for city planning…
  - when you "don't know" what is it exactly that we are looking for

[Image segmentation via clustering, James Hayes]



Clustering: basic idea

- Grouping objects into small number of meaningful groups
  - How to define similarity / distance between objects?
  - What is meaningful?
  - How many groups?
  - Typically there is no supervision



[Picture from Ackerman talk]

Developing framework: object representation

- First develop a mathematical representation of points
  - Object representation: E.g. vectors, set, sequences... when we want to represent the objects in isolation
    - Ex: Document → set / vector, image → vector , DNA → sequences
  - Interaction representation : as networks, when we are representing only the interaction between objects
    - Ex. Social / road / network,

Clustering framework: distance function

- In the object representation we need an appropriate distance function
  - Lp norms for vectors
  - Jaccard distance for sets
  - Edit distance for sequences
  - Divergences for probability distributions…

- Typically, nice to have the metric properties
  - $d(x,x) = 0,\ d(x,y) \geq 0$
  - $d(x,y) = d(y,x)$
  - $d(x,y) + d(y,z) \geq d(x,z)$

- Also nice if it is easy to calculate "average"

$$\min_x \sum_i d(p_i, x)$$

Distance function: lp norms

- L2 norm/Euclidean distance

$$D(x, y) = \sqrt[2]{\sum_{i=1}^{m} (x_i - y_i)^2}$$

- L1 norm
- L-infinity norm
- Easy to calculate averages.
- Also related is cosine distance

Objective function

- Specifying number of clusters
  - K-means / K-median
- Specifying cluster separation / quality
  - e.g. radius of cluster, Dunn's index,..
- Graph based measures
- Working w/o an objective function
  - Hierarchical clustering schemes

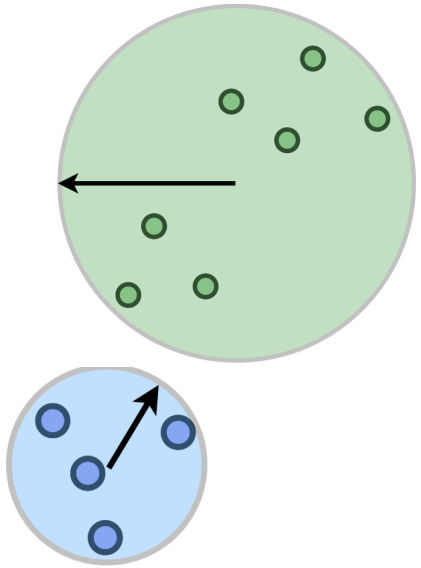
K-means

- Distance function is typically L2
- C = $\{c_1, c_2, \ldots, c_k\}$, cost(C)= $\sum_x \min_{c_x} d(x, c_x)^2$
- Find C to optimize the above cost
  - Leads to a natural partitioning of the data
- Large amount of work, both from theory & data mining community
  - Great example of divergence between theory and practice and how that prompted new research directions for both

k-means objective: alternate view

- Define "best" k-clustering of the data by
  - minimizing the "radius" of the each cluster

$$\text{minimize } \sum_i radius(C_i)$$



  - minimizing the variance of each cluster
    - The mean $c_i = \frac{1}{|C_i|}\sum_{x \in C_i} x$ is the "expected" location of a point
    - Hence variance of $C_i = \sum_{x \in C_i} ||x - c_i||^2$

[slide from Sergei V. & Suresh V.]

Lloyd’s algorithm, EM and Limitations.

The canonical algorithm: Lloyd's algorithm

- Iterative algorithm
- Iterate
  - Find current centers of partitions
  - Assign points to nearest centers
  - Recalculate centers

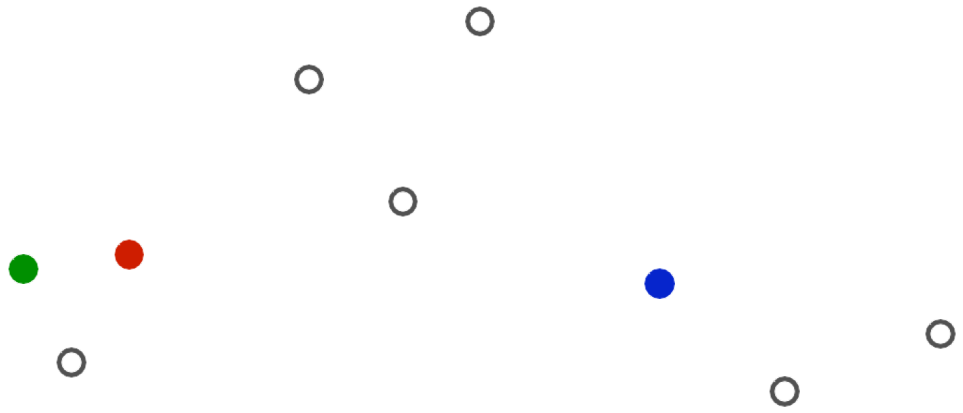
Lloyd's algorithm

- Iterative algorithm
- Iterate
  - Find current centers of partitions
  - Assign points to nearest centers
  - Recalculate centers

- Stopping criteria
  - when no (or small #) points change cluster
  - when cluster centers don't shift much
  - ….

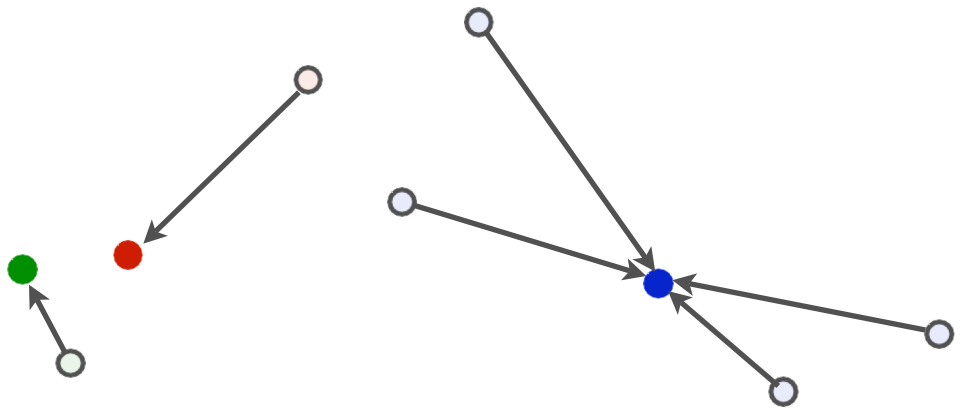
Lloyd's Method: k-means

Initialize with random clusters



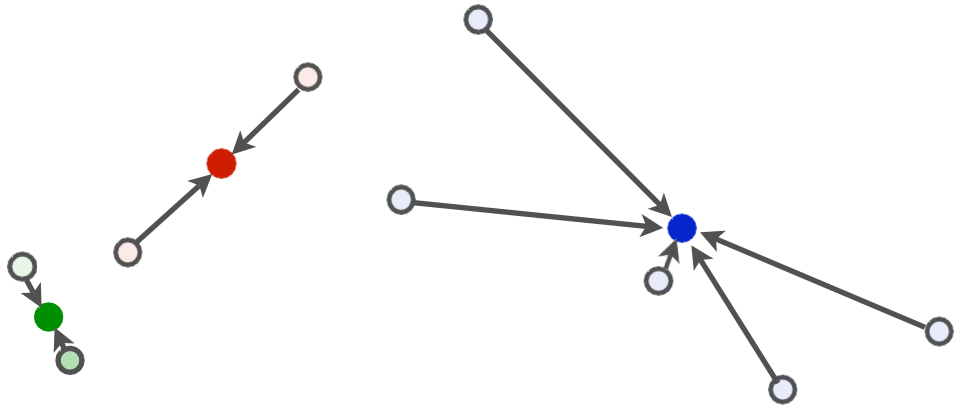
Lloyd's Method: k-means

Assign each point to nearest center



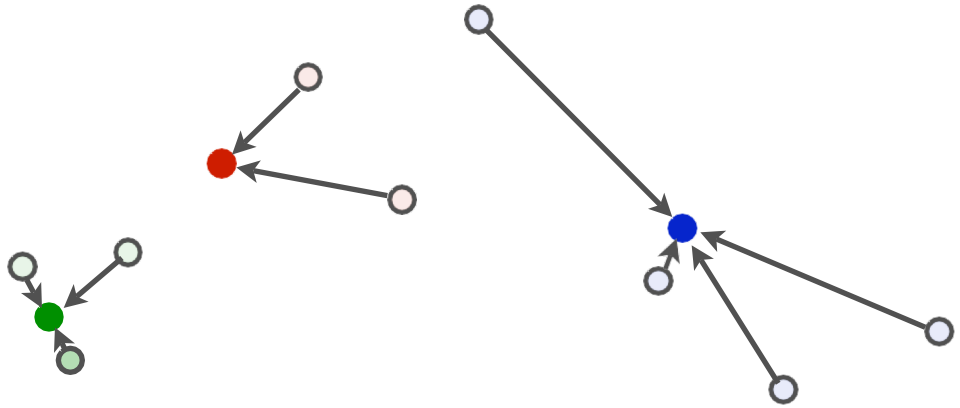
Lloyd's Method: k-means

Recompute optimum centers (means)



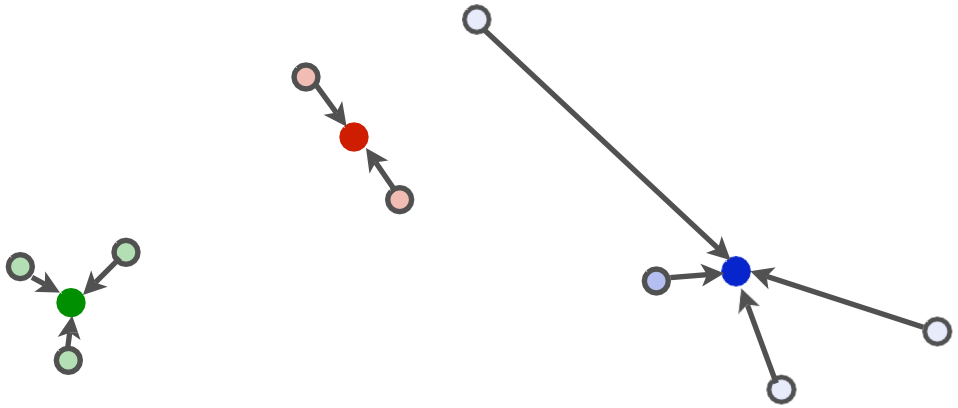
Lloyd's Method: k-means

Repeat: Assign points to nearest center



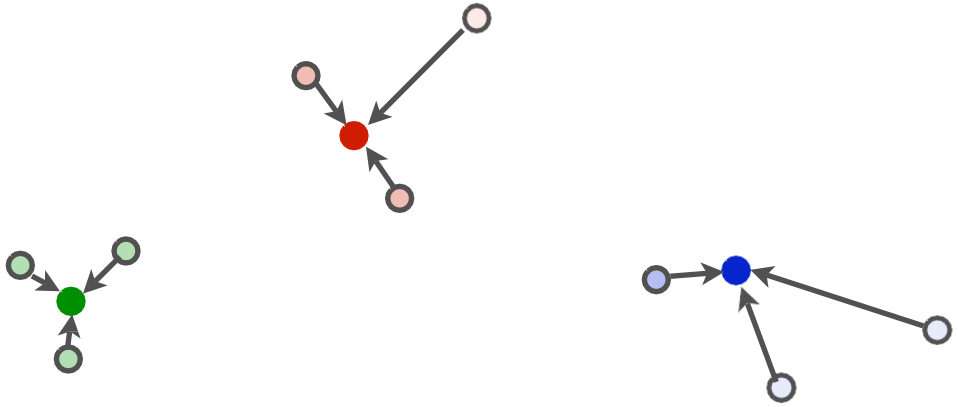
Lloyd's Method: k-means

Repeat: Recompute centers



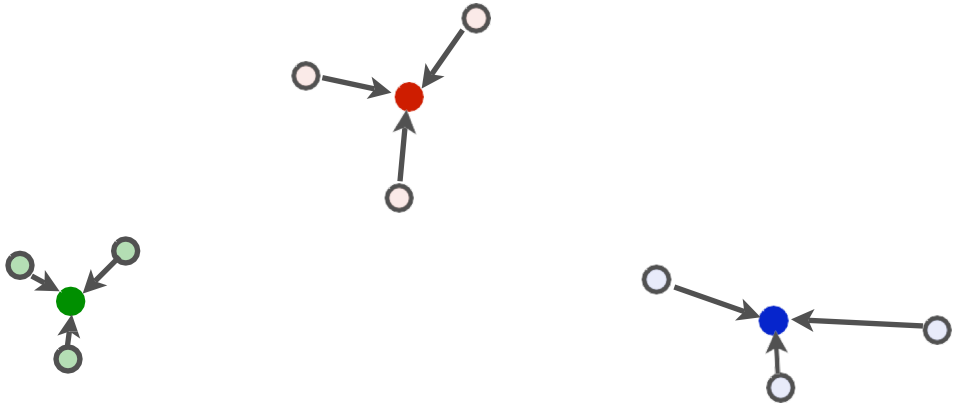
Lloyd's Method: k-means

Repeat...



Lloyd's Method: k-means

Repeat...Until clustering does not change



Lloyd's algorithm: analysis

- k centers, N points, d dimensions
- Time taken to calculate new cluster assignments : $O(k N d)$
- Time taken to calculate new centers : $O(Nd)$
- Number of iterations?

Lloyd's algorithm: convergence?

- For any current clustering, consider the objective function

$$\text{cost(C)} = \sum_x \min_{c_x} d(x, c_x)^2$$

Lloyd's algorithm: convergence?

- For any current clustering, consider the objective function

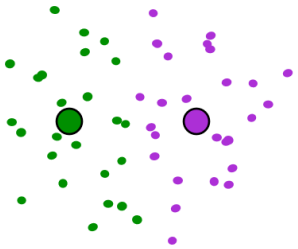
$$\text{cost(C)} = \sum_x \min_{c_x} d(x, c_x)^2$$

- At every step of the algorithm, this potentially decreases

Convergence

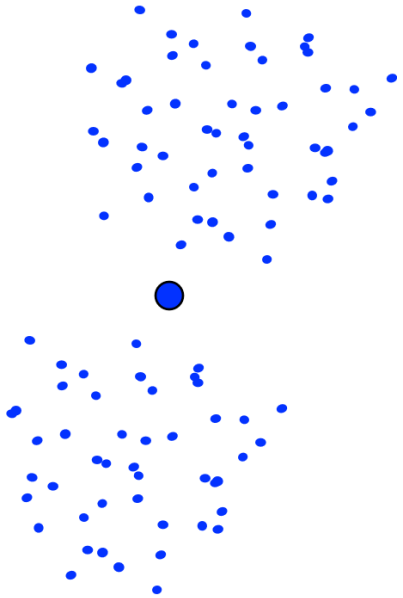
- It is known that in some datasets, Lloyd's algorithm can take exponential $(2^{\sqrt{n}})$ number of steps
  - These tend to be unrealistic

- Bigger problem is where it converges to--- depends on initialization



Should have put single cluster here

and two here

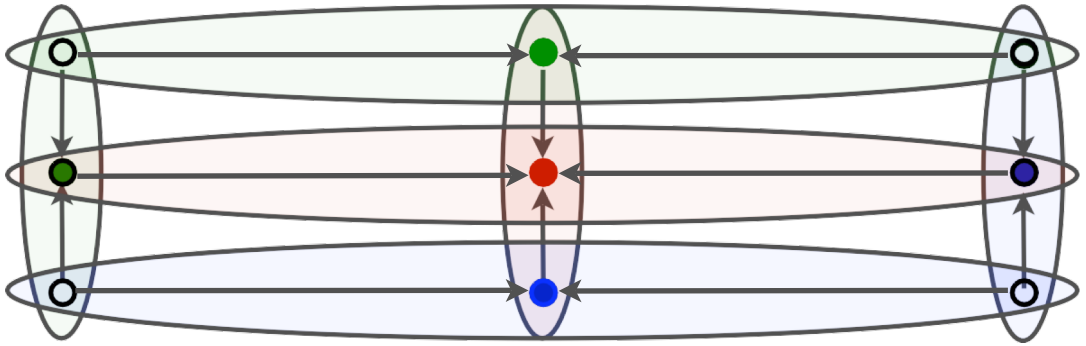


[Example from Sontag]

Convergence Analysis

Lloyd's Algorithm can be thought as a generalization of EM –algorithm for estimating mixtures of Gaussian distribution.

Finds a local optimum



That is potentially arbitrarily worse than optimal solution

K-means ++

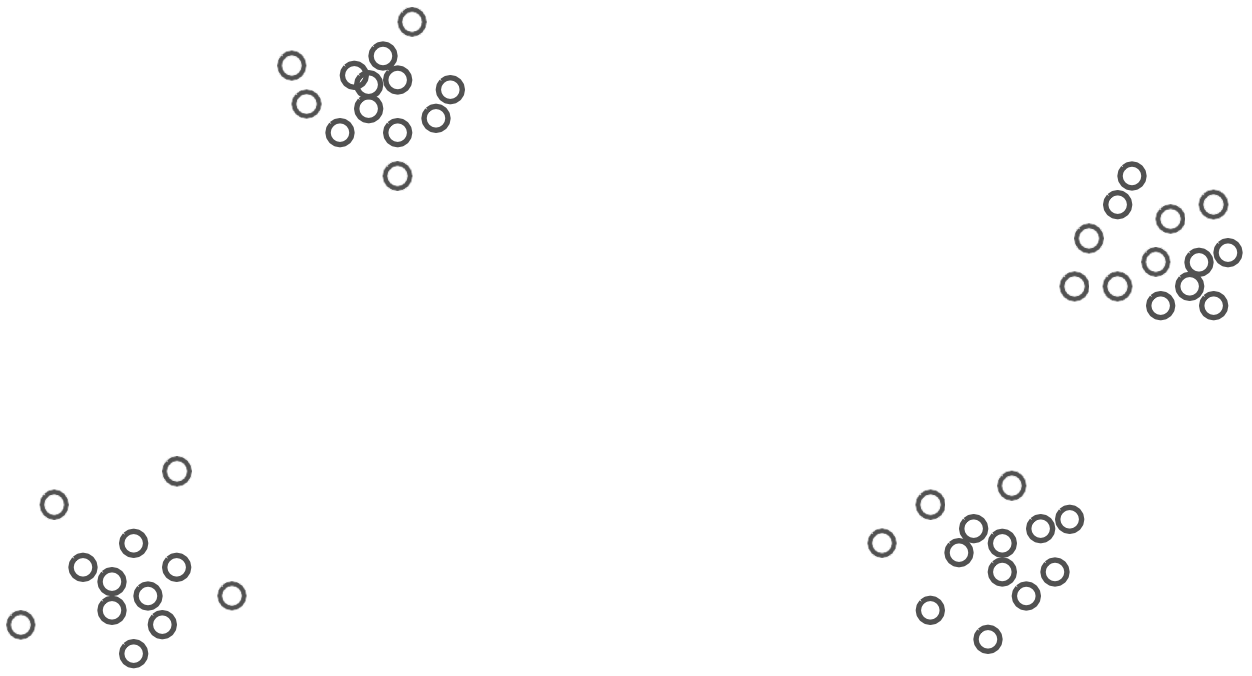
Challenge

Develop an approximation algorithm for k-means clustering that is competitive with the k-means method in speed and solution quality.

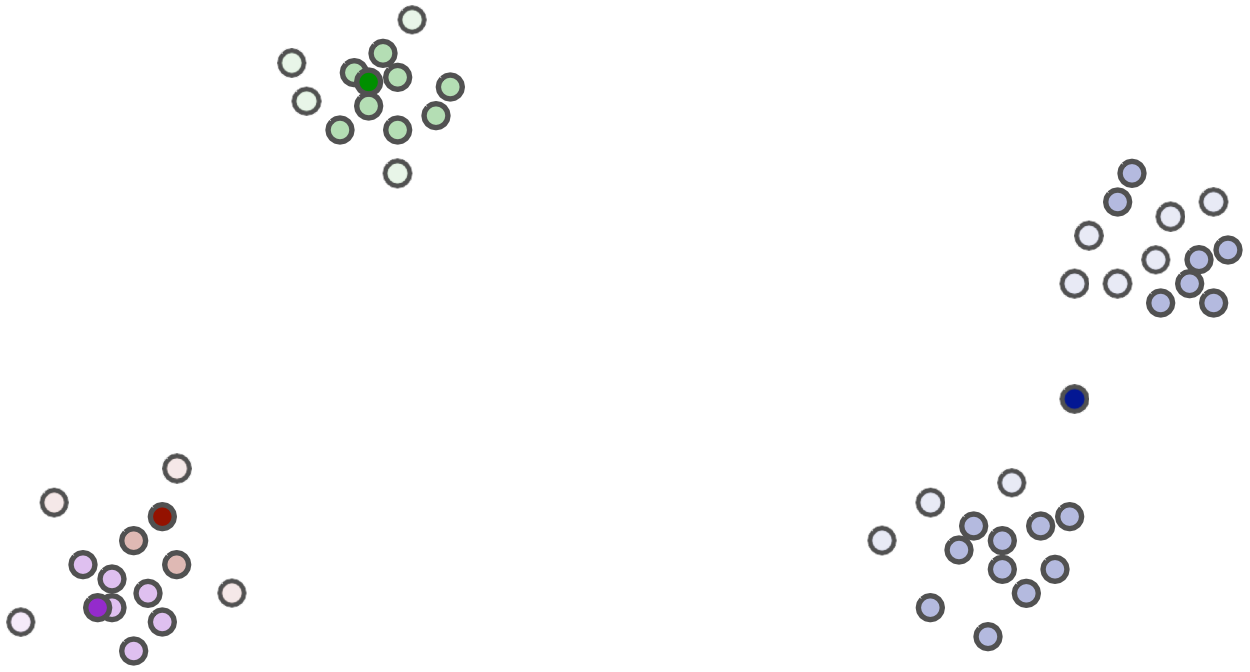
Easiest line of attack: focus on the initial center positions.

Classical k-means: pick k points at random.

k-means on Gaussians

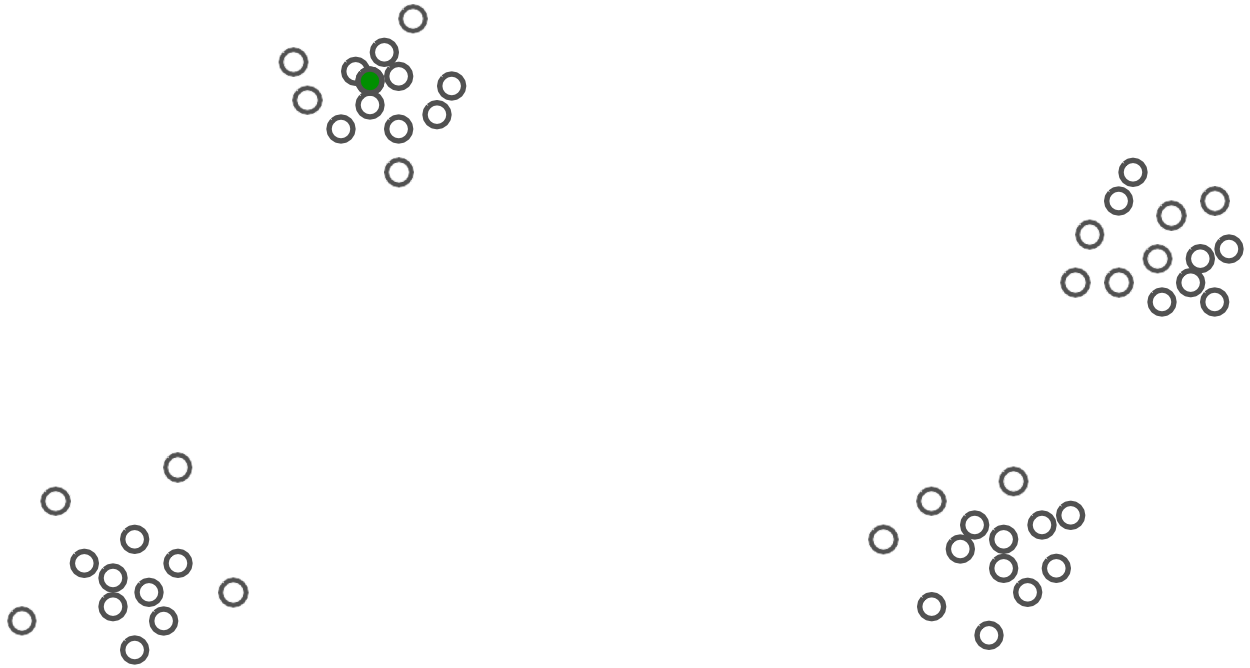


k-means on Gaussians



Easy Fix

Select centers using a furthest point algorithm (2-approximation to k-Center clustering).



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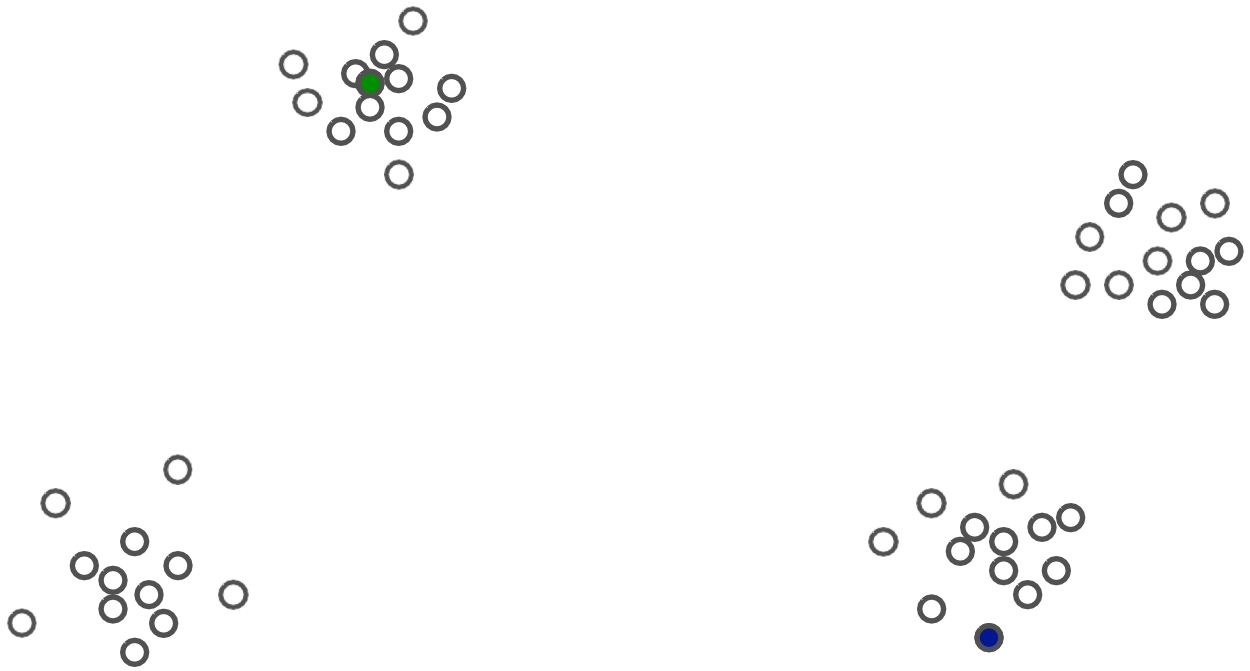

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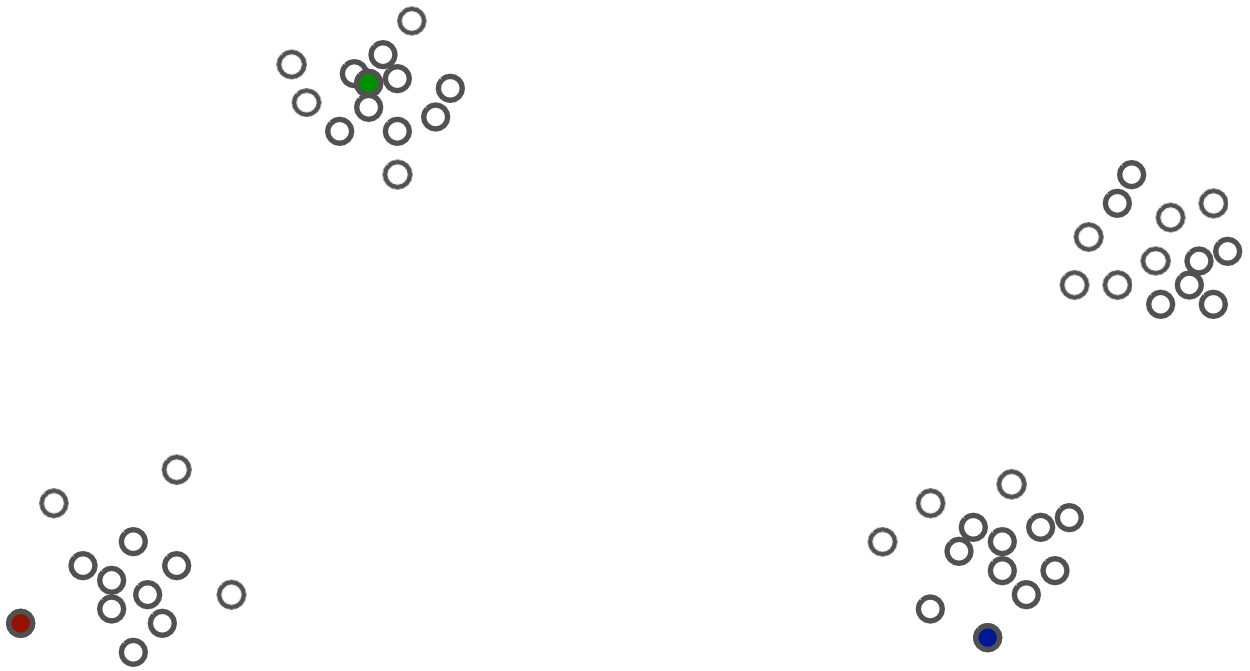

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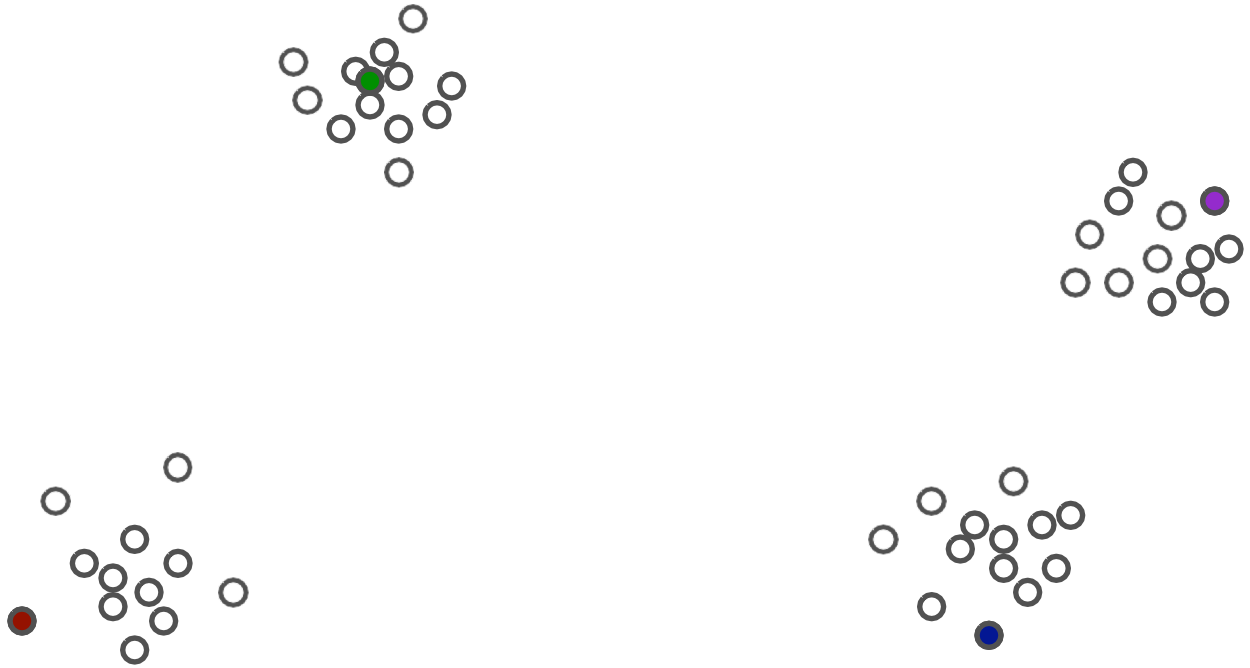

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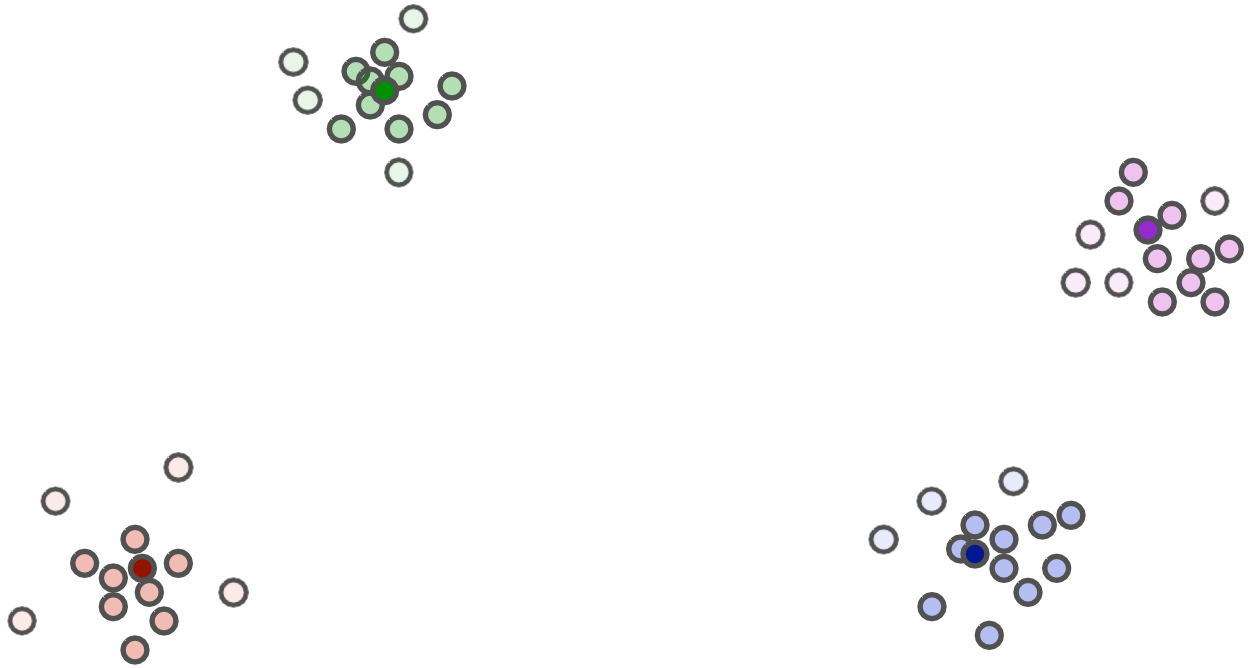
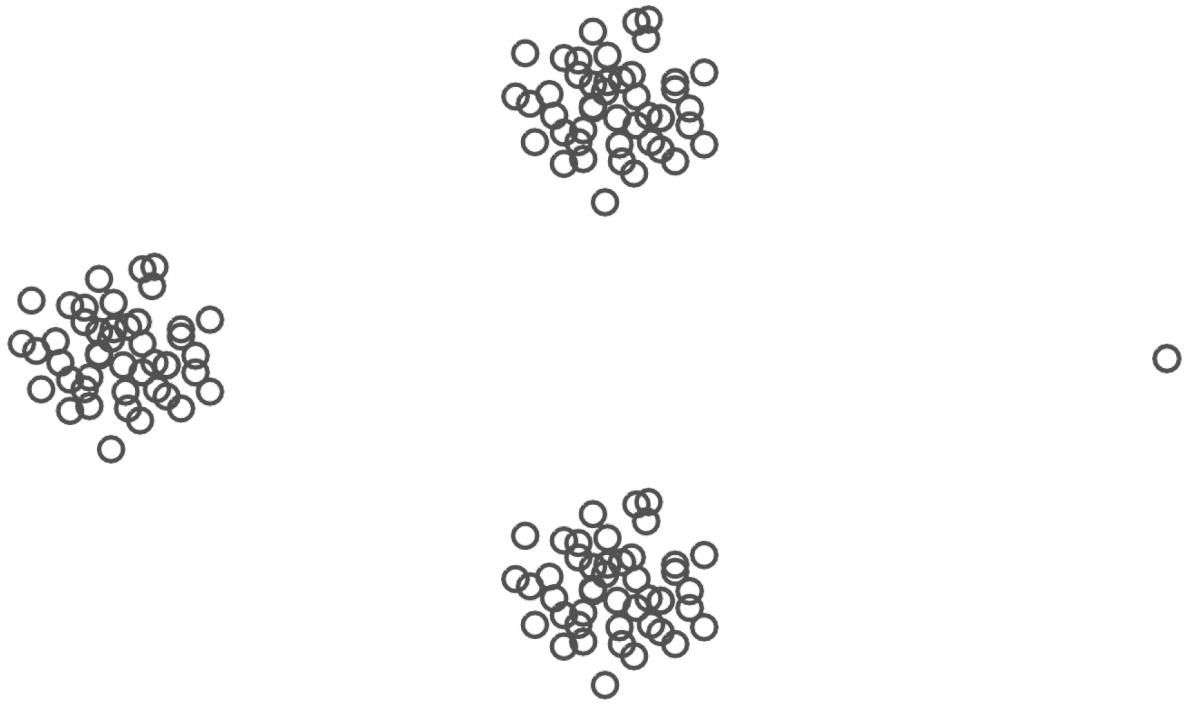


Sensitive to Outliers



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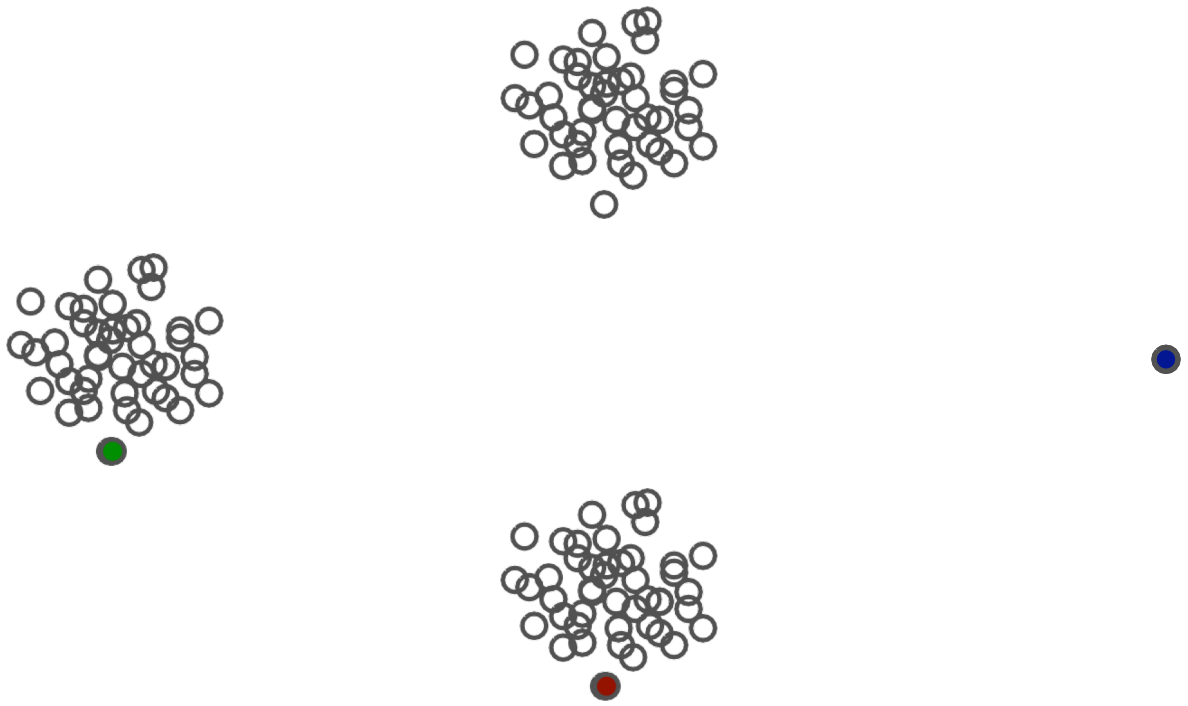

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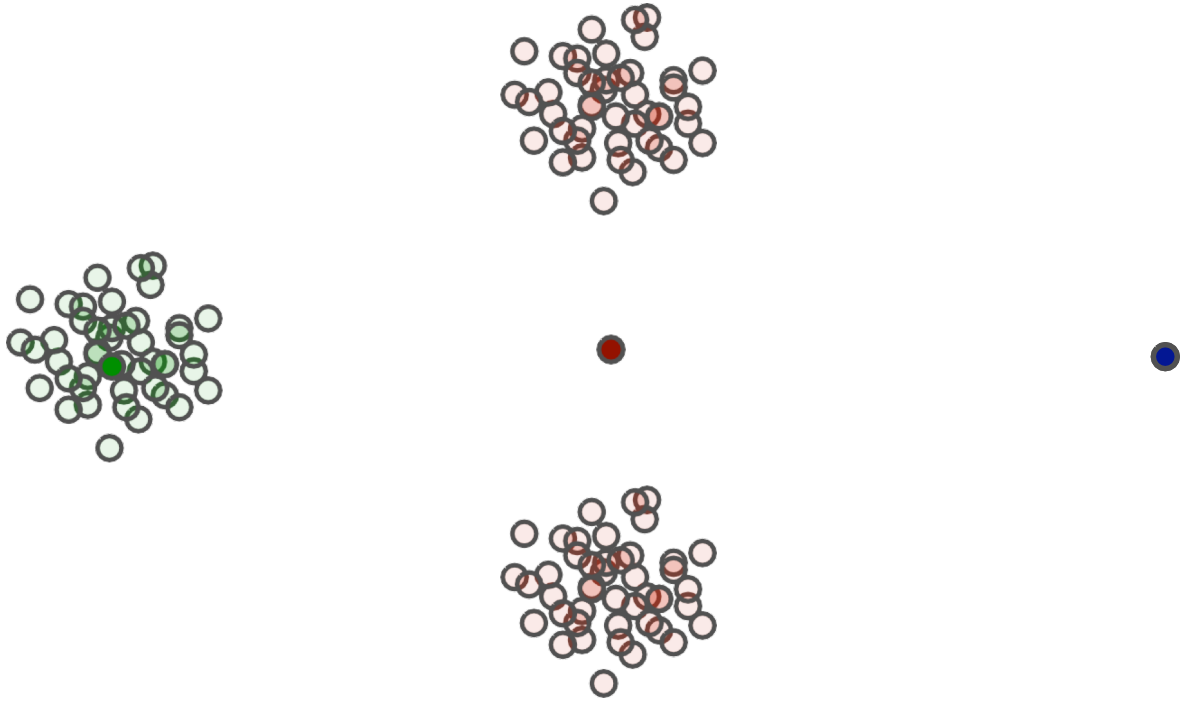


k-means++

Interpolate between the two methods:

Let $D(x)$ be the distance between x and the nearest cluster center. Sample proportionally to $(D(x))^{\alpha} = D^{\alpha}(x)$

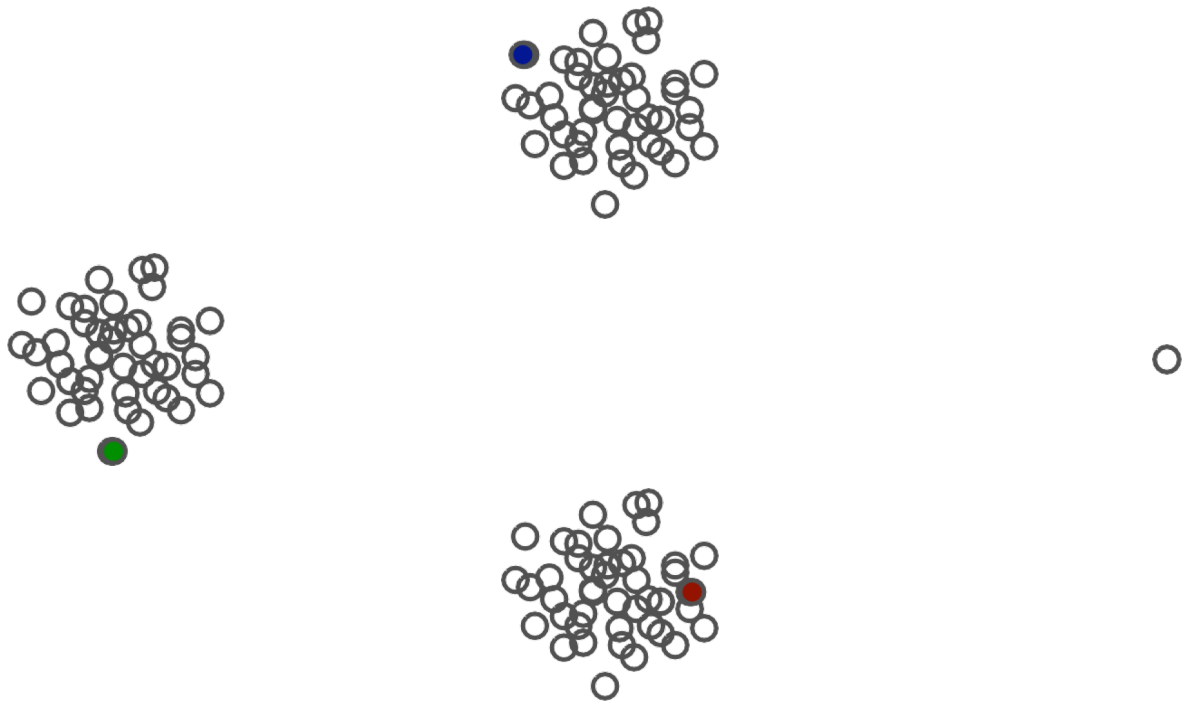
Original Lloyd's: $\alpha = 0$

Furthest Point: k-means++: $\alpha = \infty$

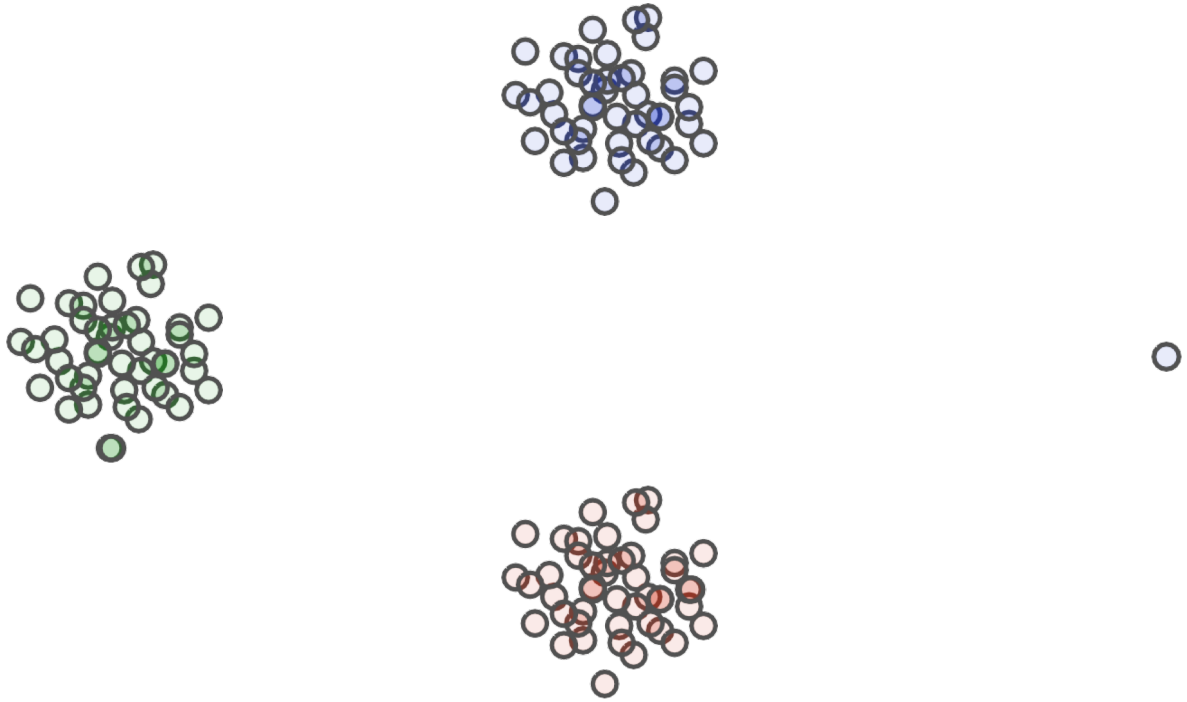
$\alpha = 2$

Contribution of x to the overall error

k-Means++



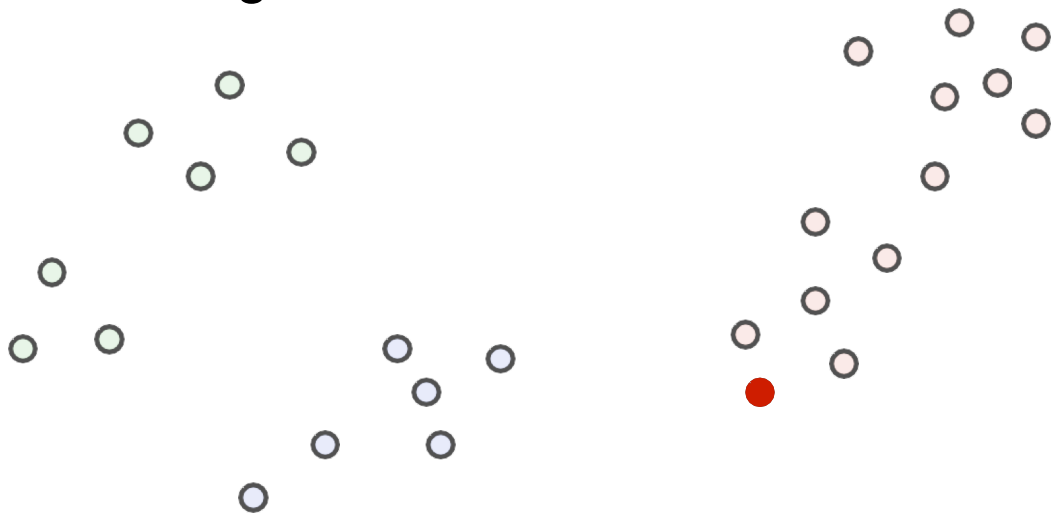
k-Means++



Theorem: k-means++ is $\Theta(\log k)$ approximate in expectation.

Proof - 1st cluster

Fix an optimal clustering C.



Pick first center uniformly at random

Bound the total error of that cluster.

Proof - 1st cluster

Let **A** be the cluster.

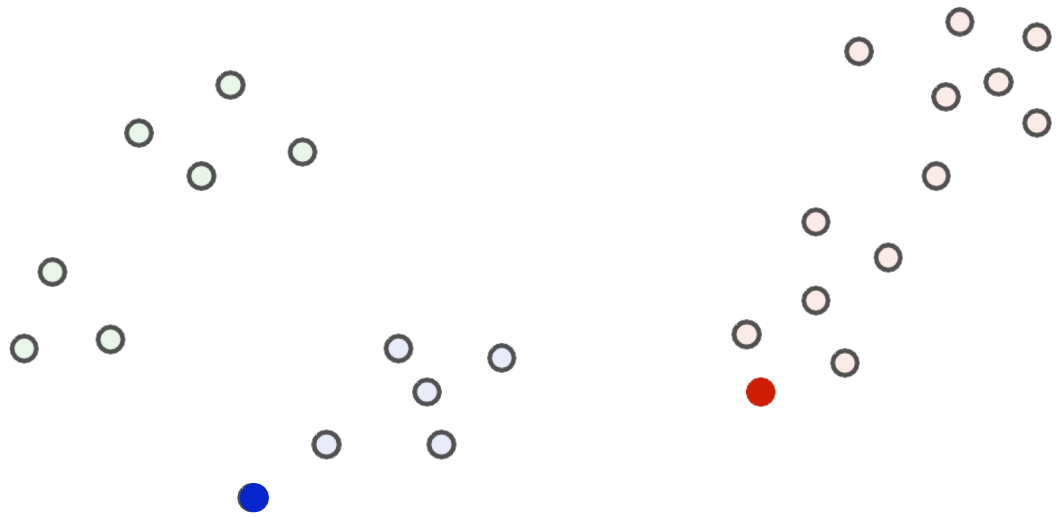
Each point $a_0 \in A$ equally likely to be the chosen center.

Expected Error:

$$E[\phi(A)] = \sum_{a_0 \in A} \frac{1}{|A|} \sum_{a \in A} \|a - a_0\|^2$$

$$= 2 \sum_{a \in A} \|a - \bar{A}\|^2 \quad = 2\phi^*(A)$$

Proof - Other Clusters



Suppose next center came from a new cluster in OPT.

Bound the total error of that cluster.

Proof - Other Clusters

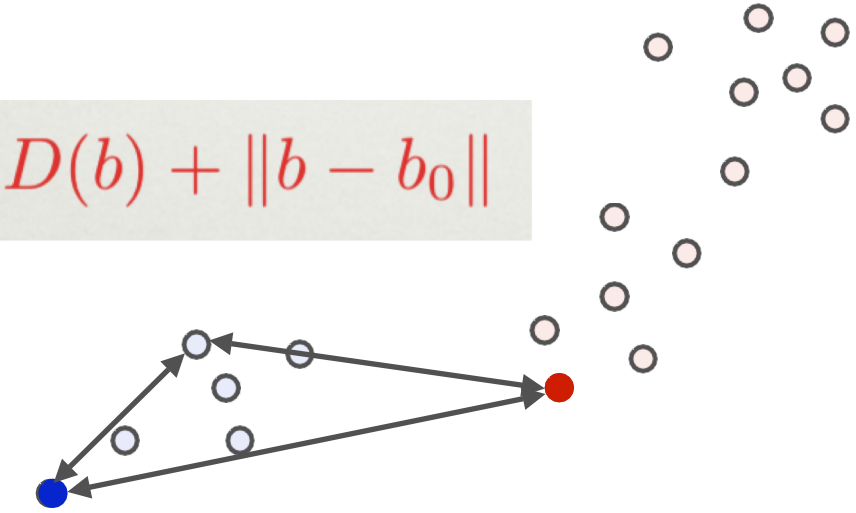
Let B be this cluster, and b_0 the point selected.

Then:

$$E[\phi(B)] = \sum_{b_0 \in B} \frac{D^2(b_0)}{\sum_{b \in B} D^2(b)} \cdot \sum_{b \in B} \min(D(b), \|b - b_0\|)^2$$

Key step:

$$D(b_0) \leq D(b) + \|b - b_0\|$$



Proof - Other Clusters

For any b: $D^2(b_0) \leq 2D^2(b) + 2\|b - b_0\|^2$

Avg. over all b: $D^2(b_0) \leq \frac{2}{|B|}\sum_{b \in B} D^2(b) + \frac{2}{|B|}\sum_{b \in B} \|b - b_0\|^2$

Same for all b_0

Cost in uniform sampling

Proof - Other Clusters

For any b: $D^2(b_0) \leq 2D^2(b) + 2\|b - b_0\|^2$

Avg. over all b: $D^2(b_0) \leq \frac{2}{|B|} \sum_{b \in B} D^2(b) + \frac{2}{|B|} \sum_{b \in B} \|b - b_0\|^2$

Recall:

$$E[\phi(B)] = \sum_{b_0 \in B} \frac{D^2(b_0)}{\sum_{b \in B} D^2(b)} \cdot \sum_{b \in B} \min(D(b), \|b - b_0\|)^2$$

$$\leq \frac{4}{|B|} \sum_{b_0 \in B} \sum_{b \in B} \|b - b_0\|^2 \quad = 8\phi^*(B)$$

Lemma – sequential uncovering

K-MEANS ||

What's wrong with K-means++?

- Needs K passes over the data
- In large data applications, not only the data is massive, but also K is typically large (e.g., easily 1000).
- Does not scale!

Intuition for a solution

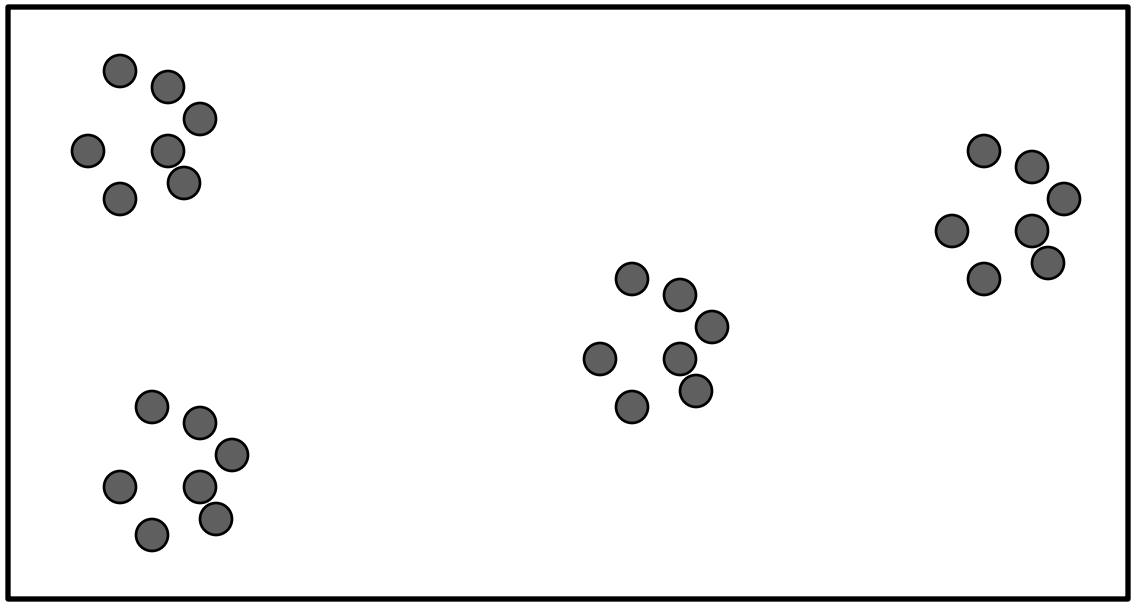
- K-means++ samples one point per iteration and updates its distribution
- What if we **oversample** by sampling each point independently with a larger probability?
- Intuitively equivalent to updating the distribution much less frequently
  - Coarser sampling
- Turns out to be sufficient: K-means||

K-means|| [Bahmani et al. ’12]

- Choose $l>1$ [Think $l=\Theta(k)$]
- Initialize C to an arbitrary set of points
- For R iterations do:
  - Sample each point x in X independently with probability $p_x = ld^2(x,C)/\varphi_X(C)$.
  - Add all the sampled points to C
- Cluster the (weighted) points in C to find the final k centers

K-means|| Initialization

K=4,
Oversampling factor =3



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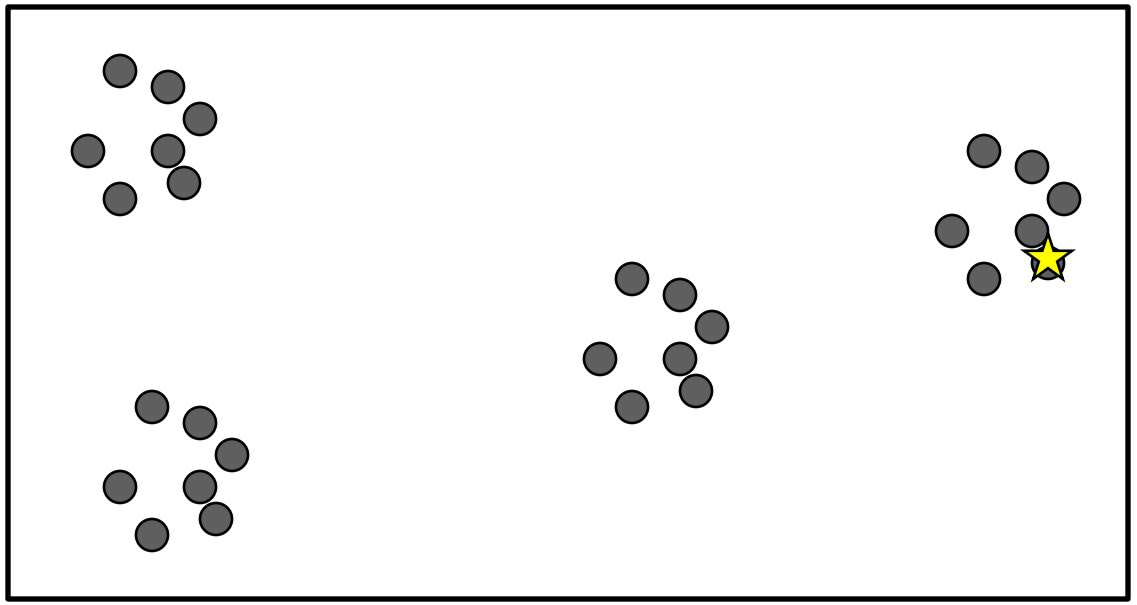

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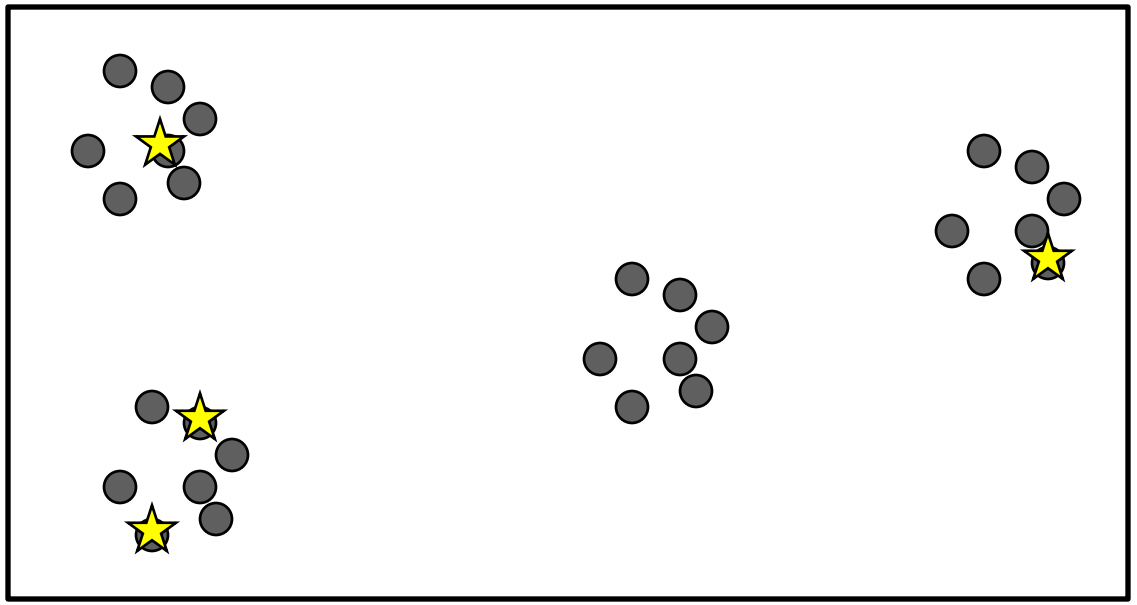

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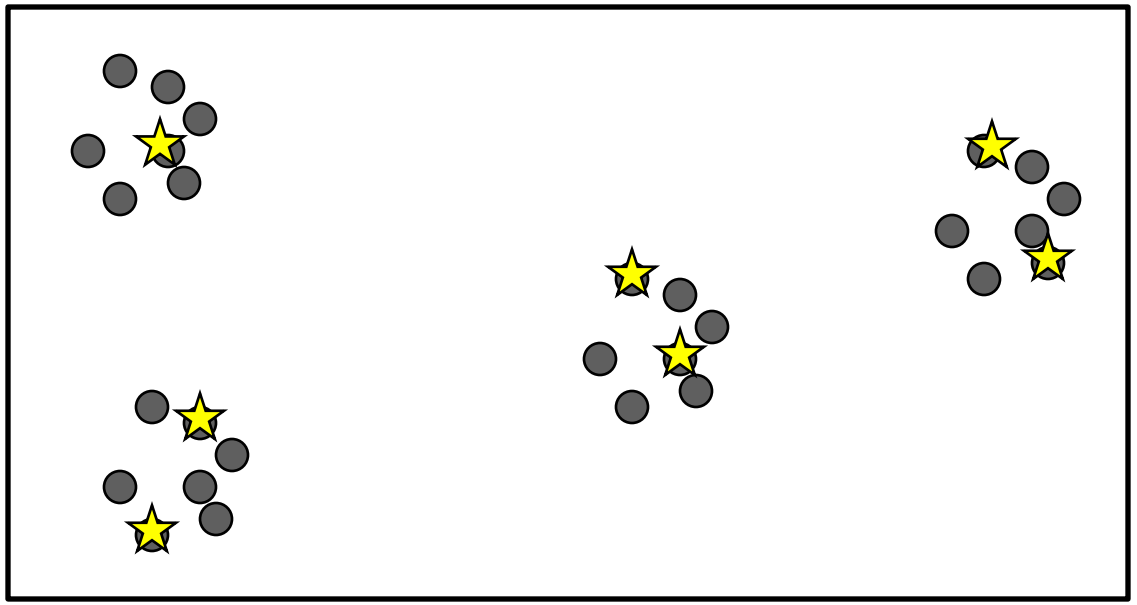

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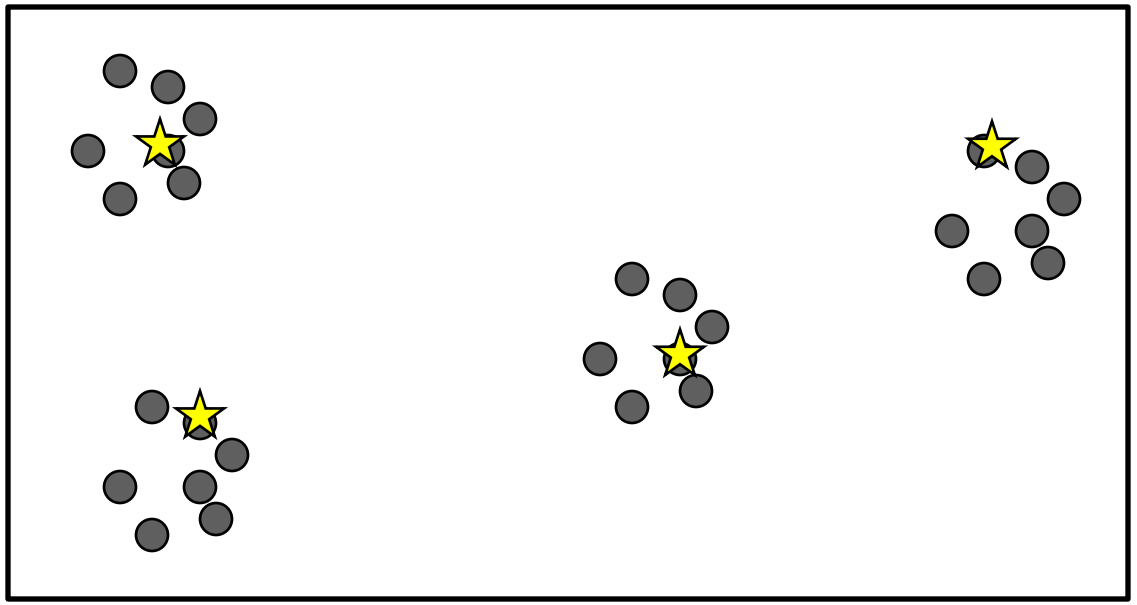

Cluster the intermediate centers

K-means|| [Bahmani et al. ’12]

- Choose $l>1$ [Think $l=\Theta(k)$]
- Initialize C to an arbitrary set of points
- For R iterations do:
  - Sample each point x in X independently with probability $p_x = l d^2(x,C)/\varphi_X(C)$.
  - Add all the sampled points to C
- Cluster the (weighted) points in C to find the final k centers

K-means||: Intuition

- An interpolation between Lloyd and K-means++

Number of iterations (R)

R=k: Simulating K-means++ (l=1) → Strong guarantee

Small R: K-means|| → **Can it possibly give any guarantees?**

R=0: Lloyd → No guarantees

Theorem

- **Theorem:** If φ and φ' are the costs of the clustering at the beginning and end of an iteration, and OPT is the cost of the optimum clustering:

$$E[\varphi'] \leq O(OPT) + \frac{k}{el}\varphi$$

- **Corollary:**
  - Let ψ= cost of initial clustering
  - K-means|| produces a constant-factor approximation to OPT, using only O(log (ψ/OPT)) iterations
  - Using K-means++ for clustering the intermediate centers, the overall approximation factor = **O(log k)**

Experimental Results: Quality

	Clustering Cost Right After Initialization	Clustering Cost After Lloyd Convergence
Random	NA	22,000
K-means++	430	65
K-means||	16	14

GAUSSMIXTURE: 10,000 points in 15 dimensions
K=50
Costs scaled down by 10^4

- K-means|| much harder than K-means++ to get confused with noisy outliers

Experimental Results: Convergence

- K-means|| reduces number of Lloyd iterations even more than K-means++

	Number of Lloyd Iterations till Convergence
Random	167
K-means++	42
K-means||	28

SPAM: 4,601 points in 58 dimensions
K=50

Experimental Results

- K-means|| needs a small number of intermediate centers
- Better than K-means++ as soon as ~K centers chosen

	Clustering Cost (Scaled down by 10^{10})	Number of intermediate centers	Tme (In Minutes)
Random	$6.4 * 10^7$	NA	489
Partition	1.9	$1.47 * 10^6$	1022
K-means||	1.5	3604	87

KDDCUP1999: 4.8M points in 42 dimensions
K=1000

Algorithmic Theme

- Quickly decrease the size of the data in a distributed fashion…
- … while maintaining the important features of the data
- Solve the small instance on a single machine

Thank You!!

Faculty Name
Department Name