CS60020: Foundations of Algorithm Design and Machine Learning

Sourangshu Bhattacharya

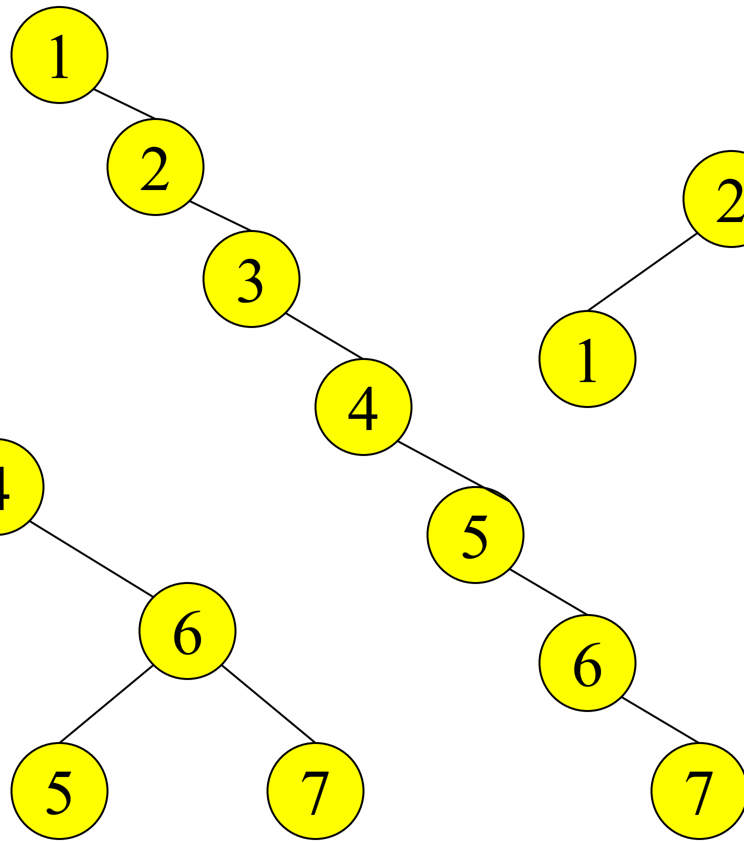
Binary Search Tree - Best Time

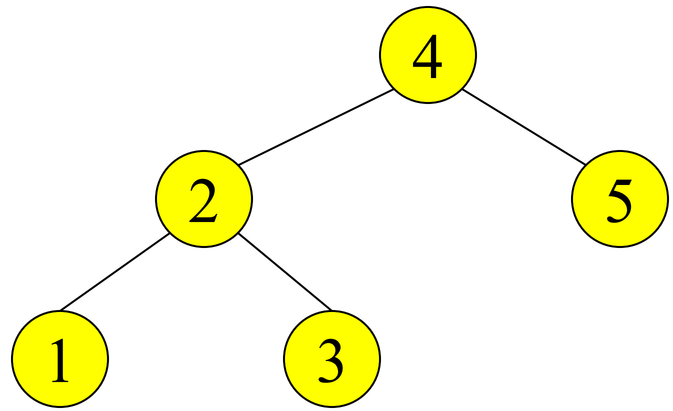
- All BST operations are O(d), where d is tree depth
- minimum d is $d = \lfloor \log_2 N \rfloor$ for a binary tree with N nodes
  - What is the best case tree?
  - What is the worst case tree?
- So, best case running time of BST operations is O(log N)

Binary Search Tree - Worst Time

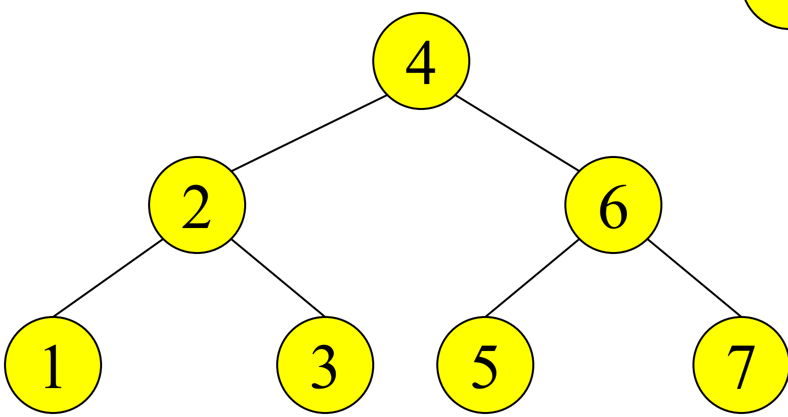
- Worst case running time is O(N)
  - › What happens when you Insert elements in ascending order?
    - Insert: 2, 4, 6, 8, 10, 12 into an empty BST
  - › Problem: Lack of "balance":
    - compare depths of left and right subtree
  - › Unbalanced degenerate tree

Balanced and unbalanced BST





Is this “balanced”?



Approaches to balancing trees

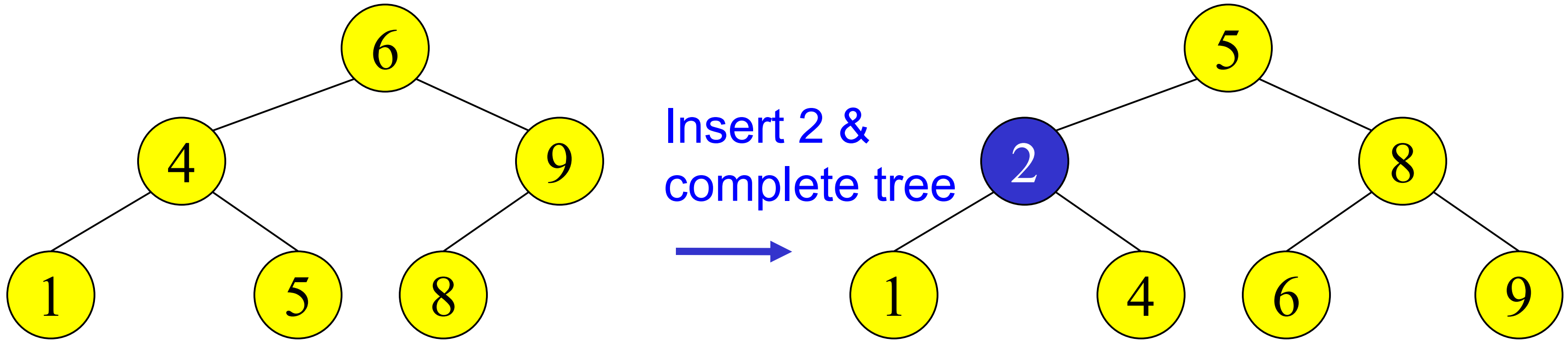
- Don't balance
  - May end up with some nodes very deep
- Strict balance
  - The tree must always be balanced perfectly
- Pretty good balance
  - Only allow a little out of balance
- Adjust on access
  - Self-adjusting

Balancing Binary Search Trees

- Many algorithms exist for keeping binary search trees balanced
  - Adelson-Velskii and Landis (AVL) trees (height-balanced trees)
  - Splay trees and other self-adjusting trees
  - B-trees and other multiway search trees

Perfect Balance

- Want a complete tree after every operation
  - tree is full except possibly in the lower right
- This is expensive
  - For example, insert 2 in the tree on the left and then rebuild as a complete tree

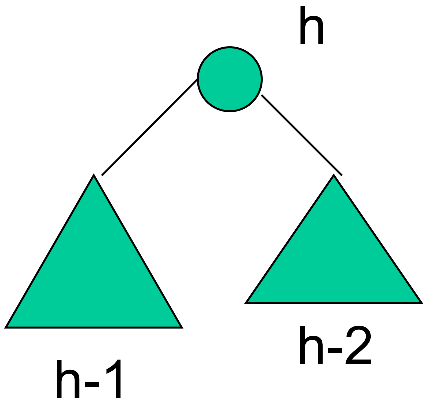


AVL - Good but not Perfect Balance

- AVL trees are height-balanced binary search trees
- Balance factor of a node
  - height(left subtree) - height(right subtree)
- An AVL tree has balance factor calculated at every node
  - For every node, heights of left and right subtree can differ by no more than 1
  - Store current heights in each node

Height of an AVL Tree

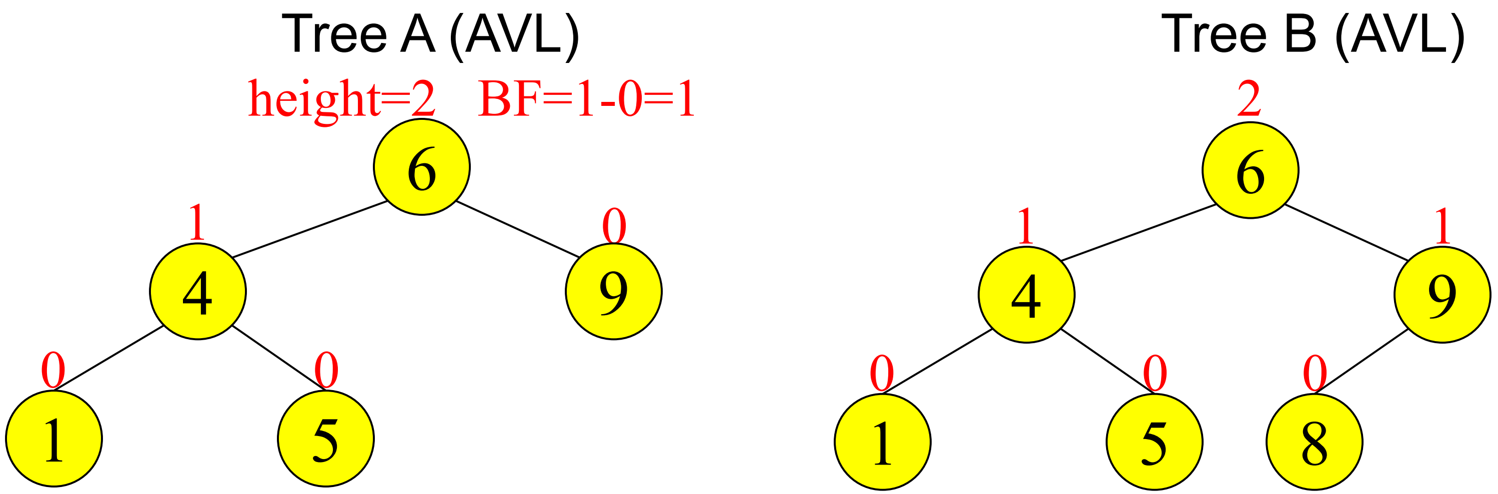
- $N(h)$ = minimum number of nodes in an AVL tree of height h.
- Basis
  - › $N(0) = 1$, $N(1) = 2$
- Induction
  - › $N(h) = N(h-1) + N(h-2) + 1$
- Solution (recall Fibonacci analysis)
  - › $N(h) \geq \phi^h \quad (\phi \approx 1.62)$



Height of an AVL Tree

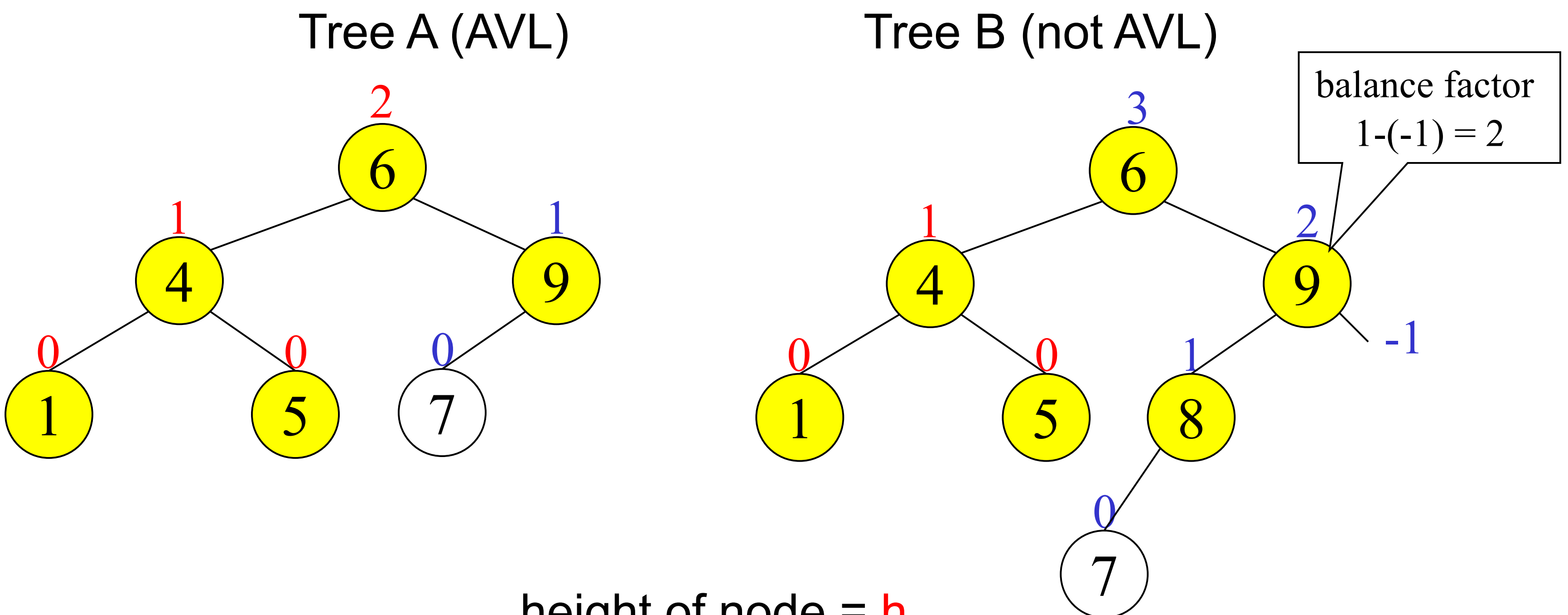
- $N(h) \geq \phi^h$ ($\phi \approx 1.62$)
- Suppose we have n nodes in an AVL tree of height h.
  - $n \geq N(h)$ (because N(h) was the minimum)
  - $n \geq \phi^h$ hence $\log_\phi n \geq h$ (relatively well balanced tree!!)
  - $h \leq 1.44 \log_2 n$ (i.e., Find takes O(log n))

Node Heights



height of node = h
balance factor = $h_{left}-h_{right}$
empty height = -1

Node Heights after Insert 7

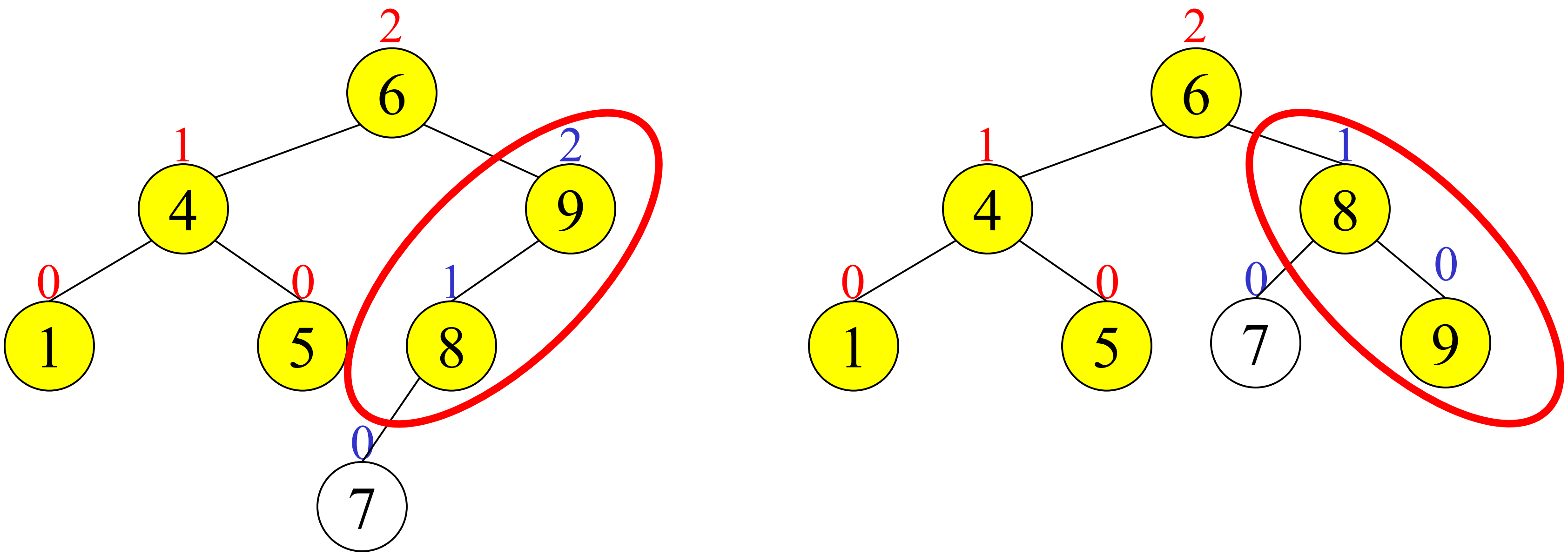


height of node = h
balance factor = h_{left}-h_{right}
empty height = -1

Insert and Rotation in AVL Trees

- Insert operation may cause balance factor to become 2 or –2 for some node
  - only nodes on the path from insertion point to root node have possibly changed in height
  - So after the Insert, go back up to the root node by node, updating heights
  - If a new balance factor (the difference h_{left}-h_{right}) is 2 or –2, adjust tree by *rotation* around the node

Single Rotation in an AVL Tree



Insertions in AVL Trees

Let the node that needs rebalancing be α.

There are 4 cases:

Outside Cases (require single rotation) :

1. Insertion into left subtree of left child of α.
2. Insertion into right subtree of right child of α.

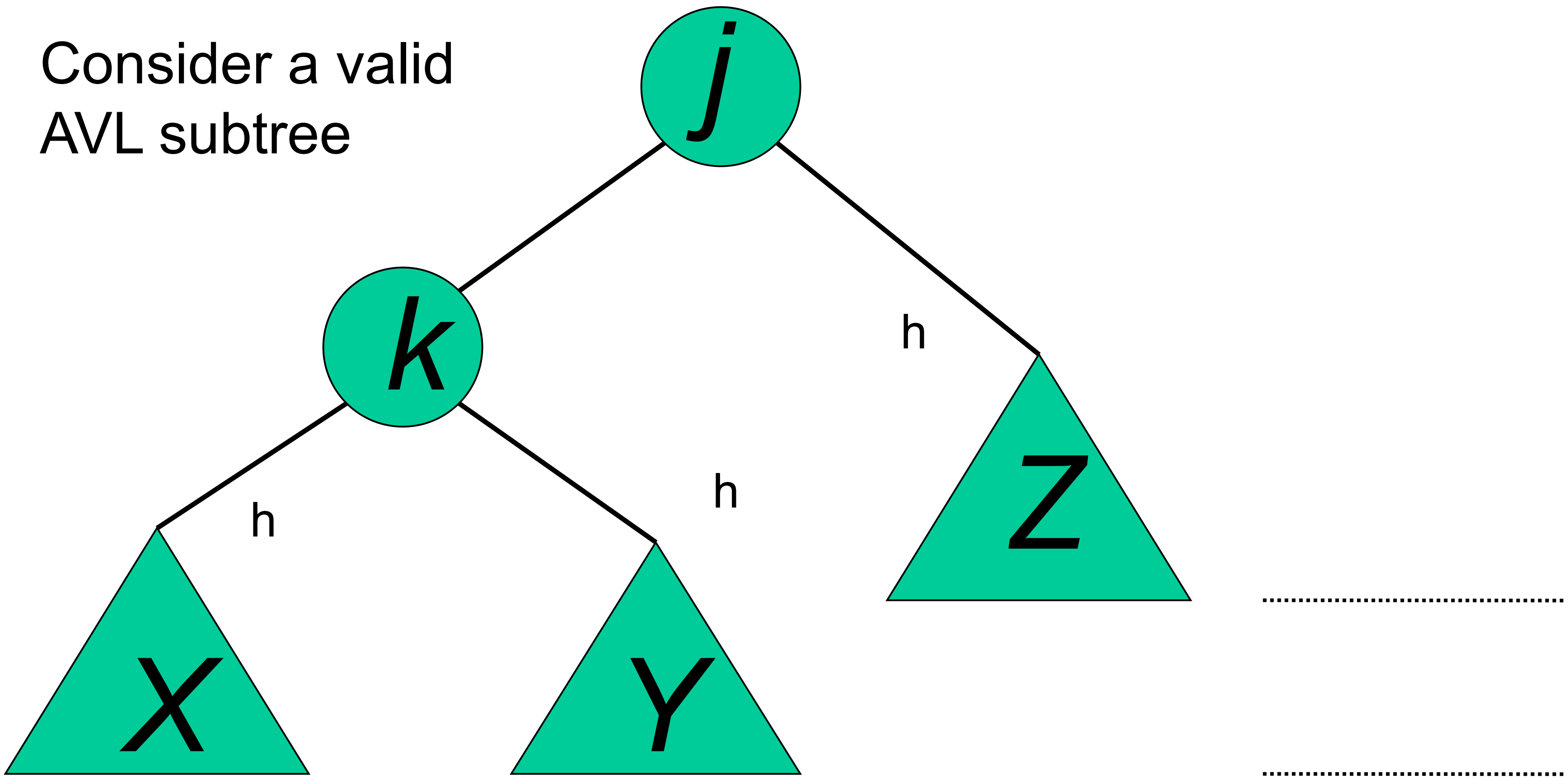
Inside Cases (require double rotation) :

3. Insertion into right subtree of left child of α.
4. Insertion into left subtree of right child of α.

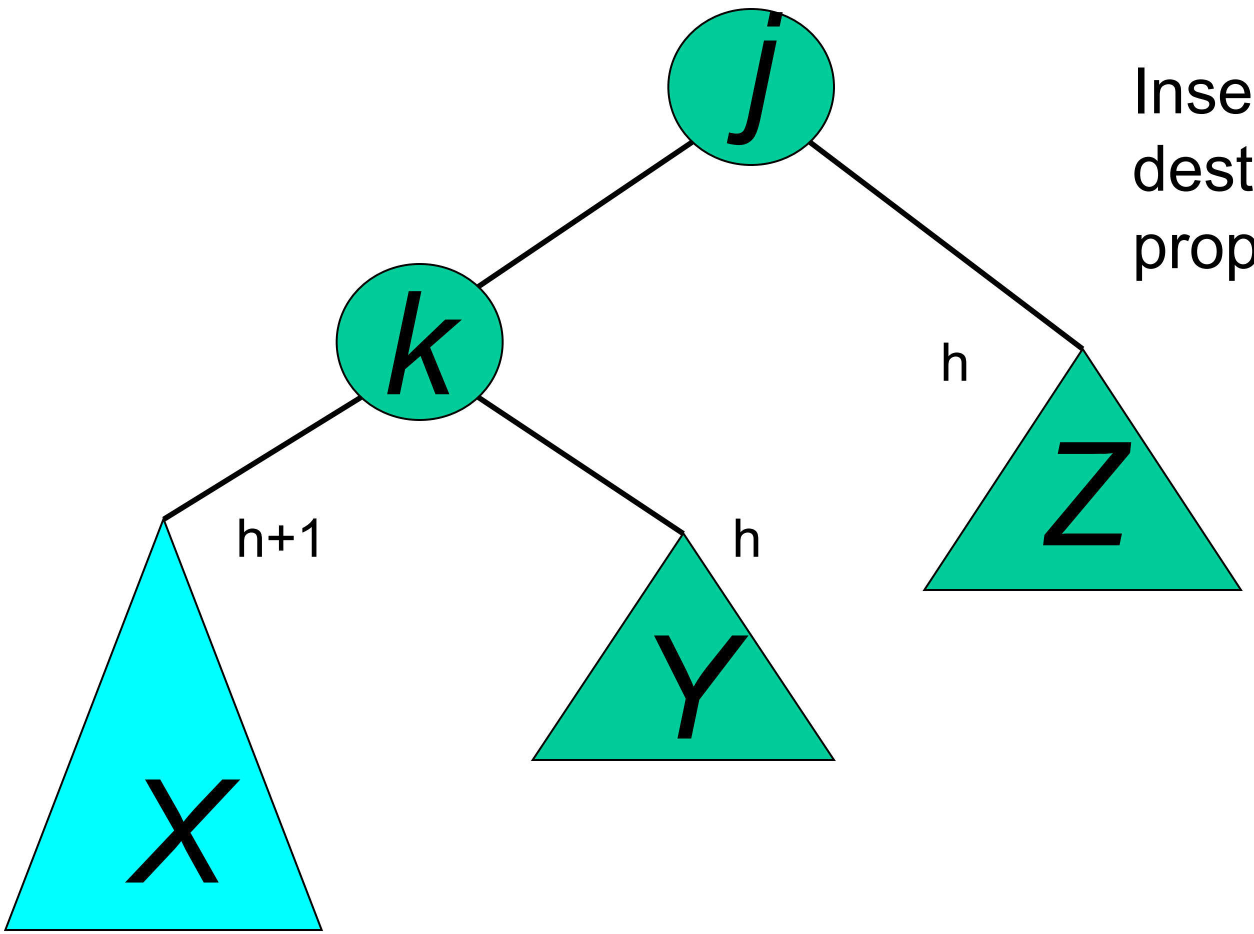
The rebalancing is performed through four separate rotation algorithms.

AVL Insertion: Outside Case

Consider a valid
AVL subtree

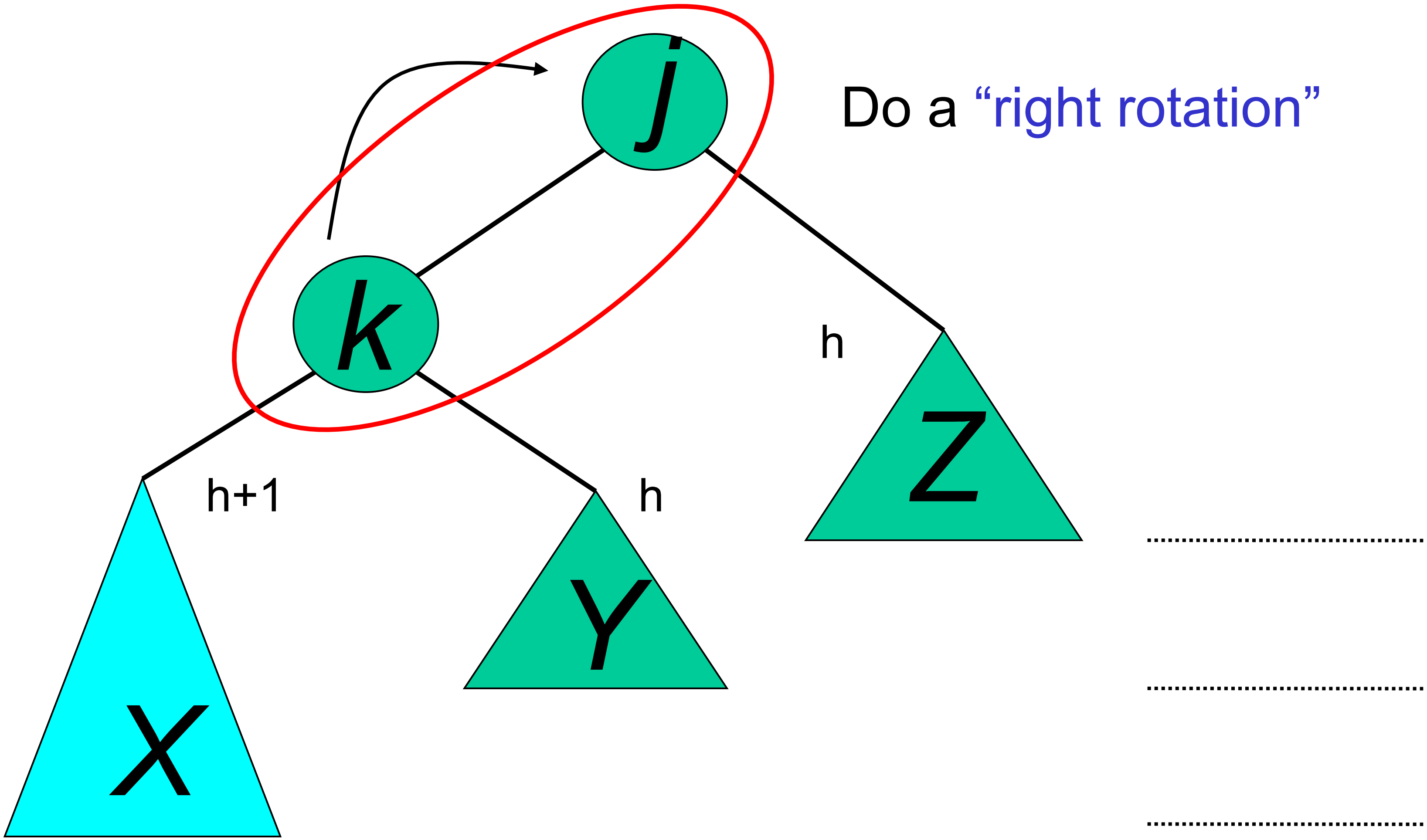


AVL Insertion: Outside Case



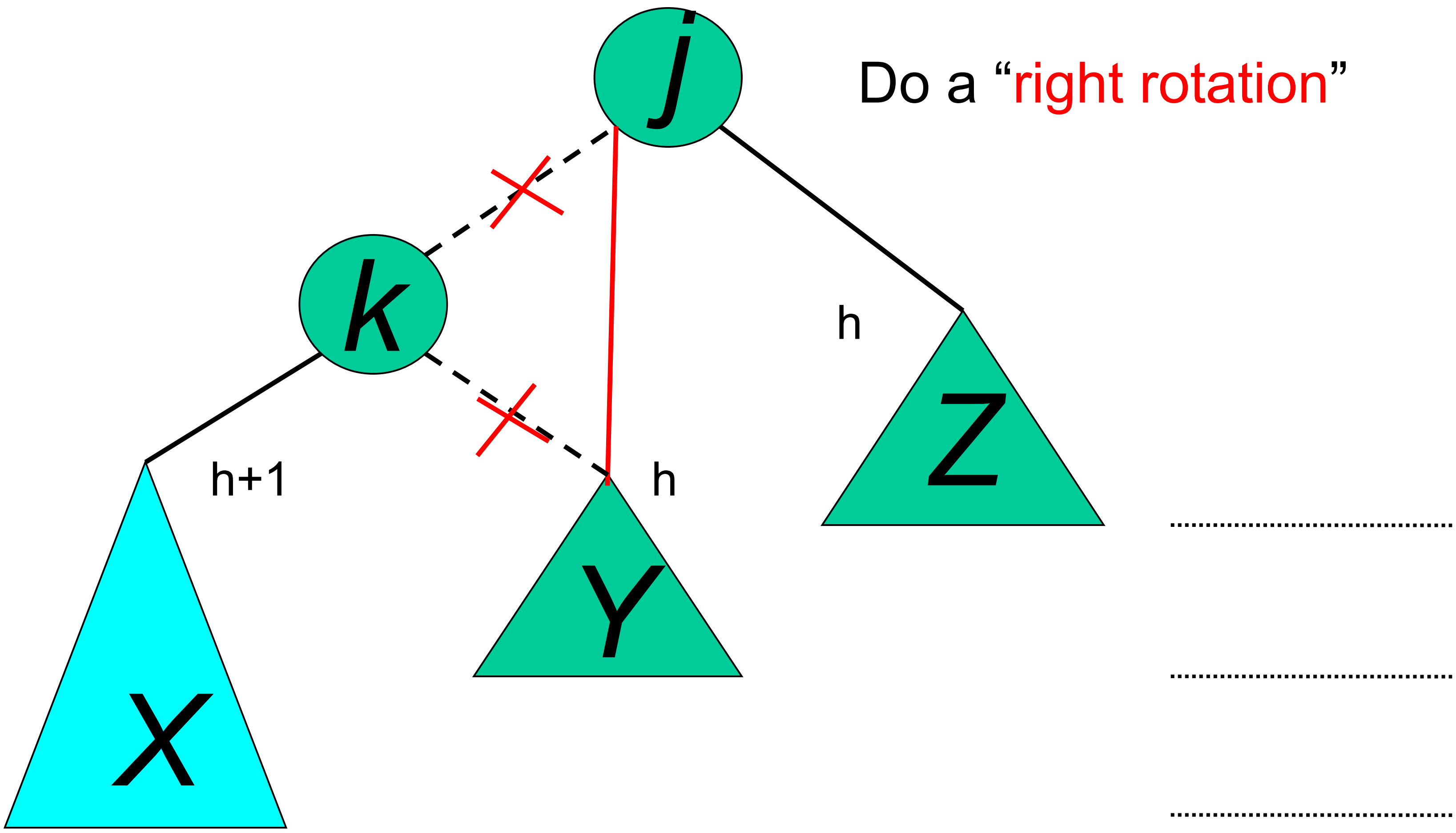
Inserting into X destroys the AVL property at node j

AVL Insertion: Outside Case



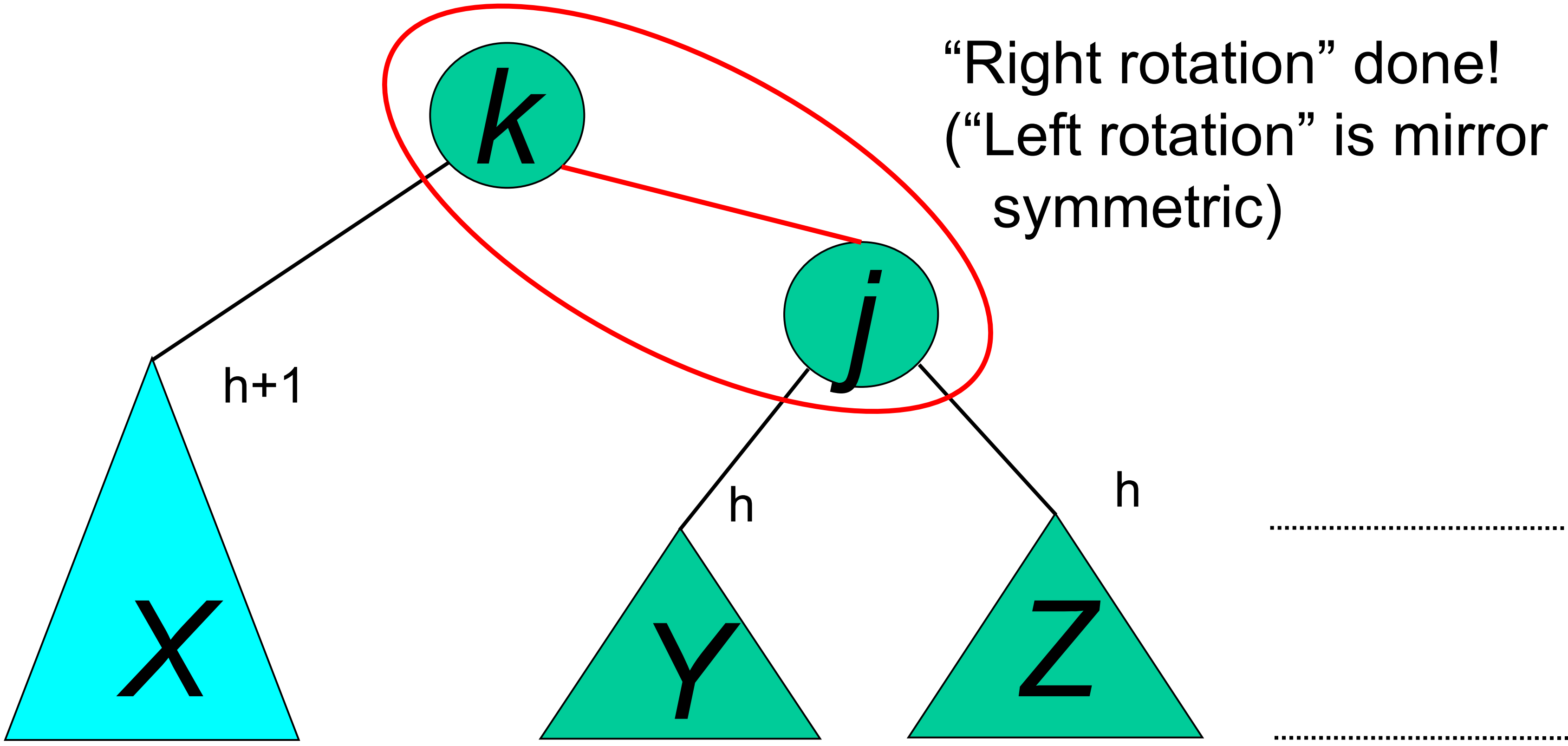
Do a "right rotation"

Single right rotation



Do a "right rotation"

Outside Case Completed

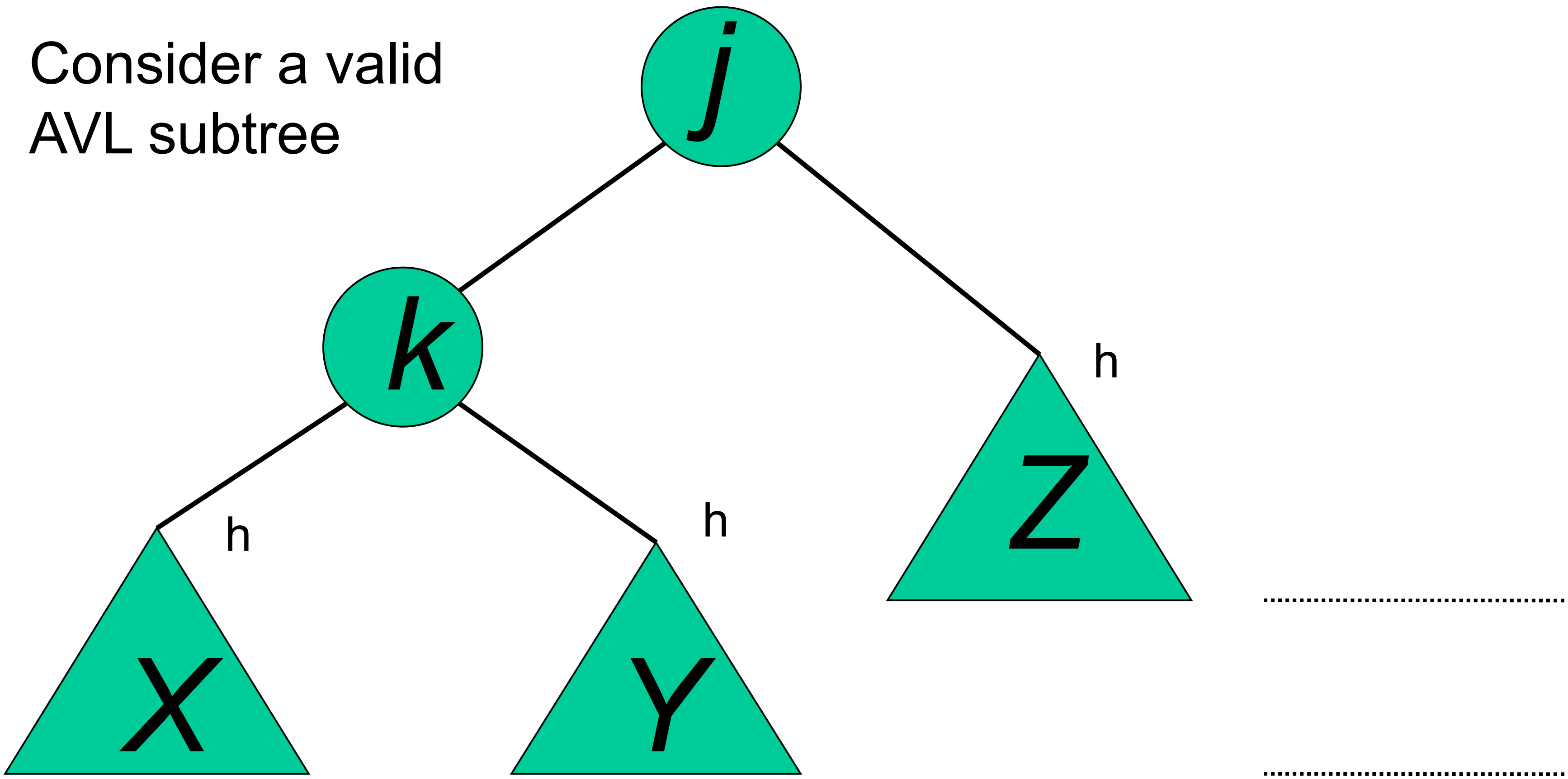


"Right rotation" done!
("Left rotation" is mirror symmetric)

AVL property has been restored!

AVL Insertion: Inside Case

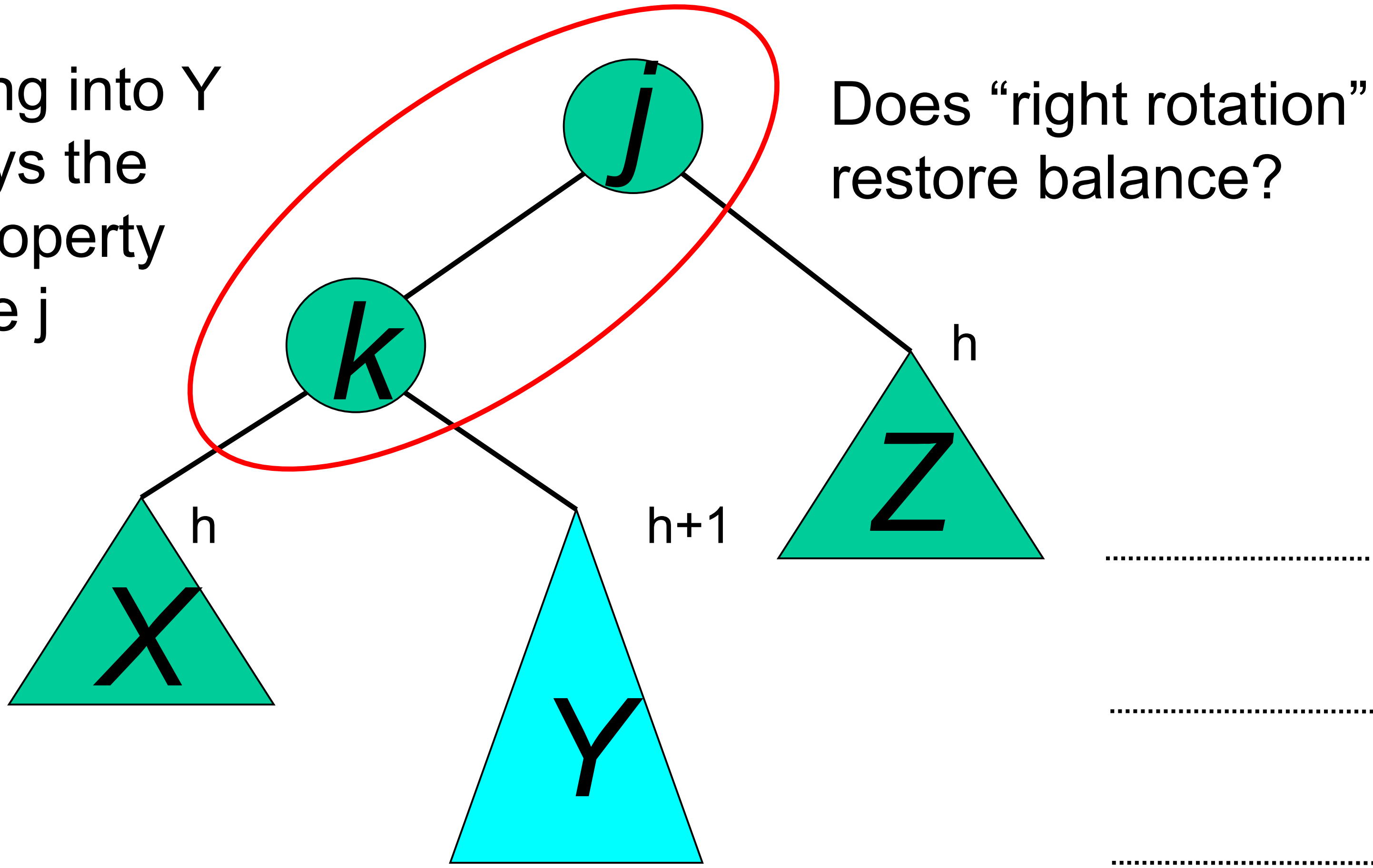
Consider a valid AVL subtree



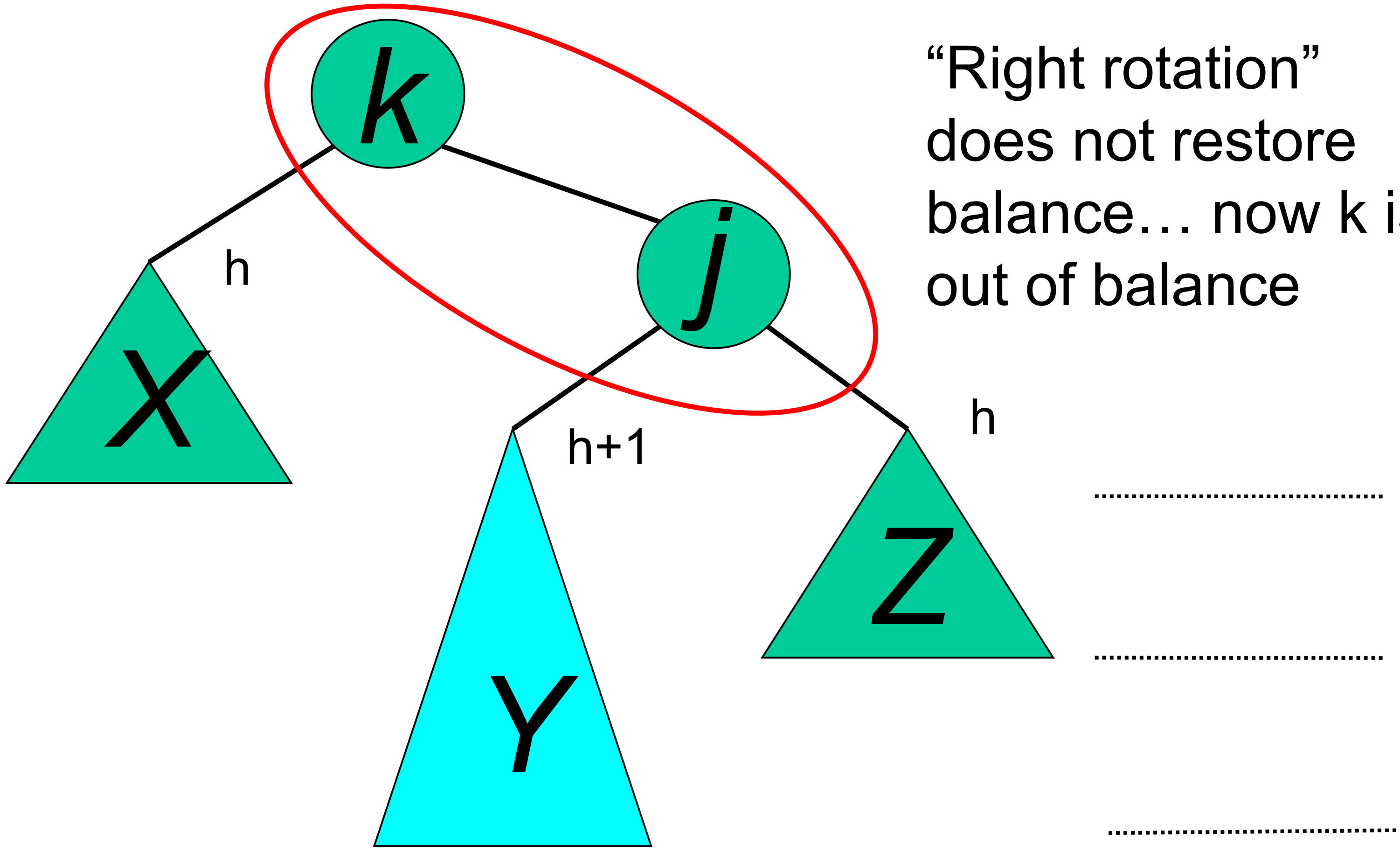
AVL Insertion: Inside Case

Inserting into Y destroys the AVL property at node j

Does "right rotation" restore balance?



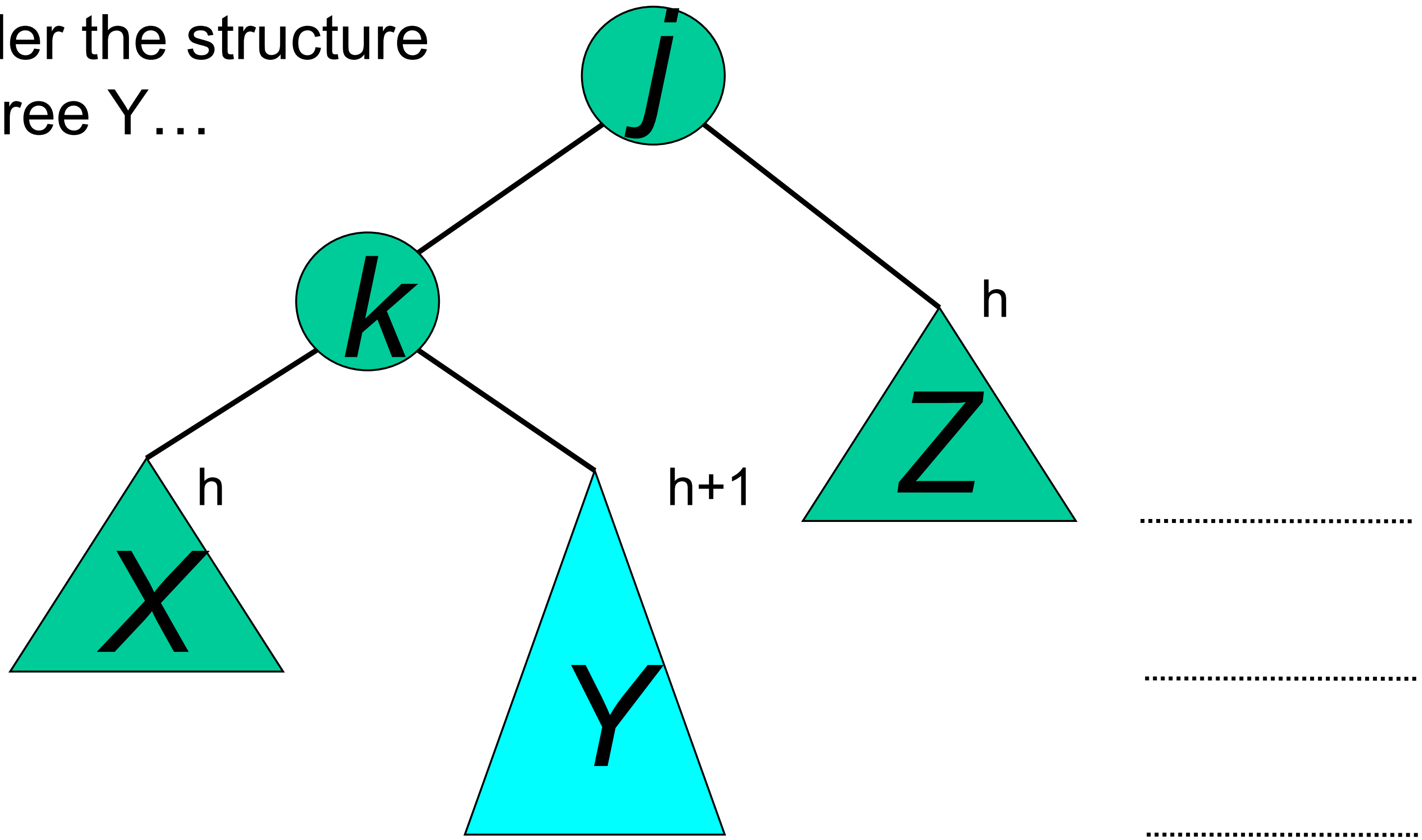
AVL Insertion: Inside Case



"Right rotation" does not restore balance… now k is out of balance

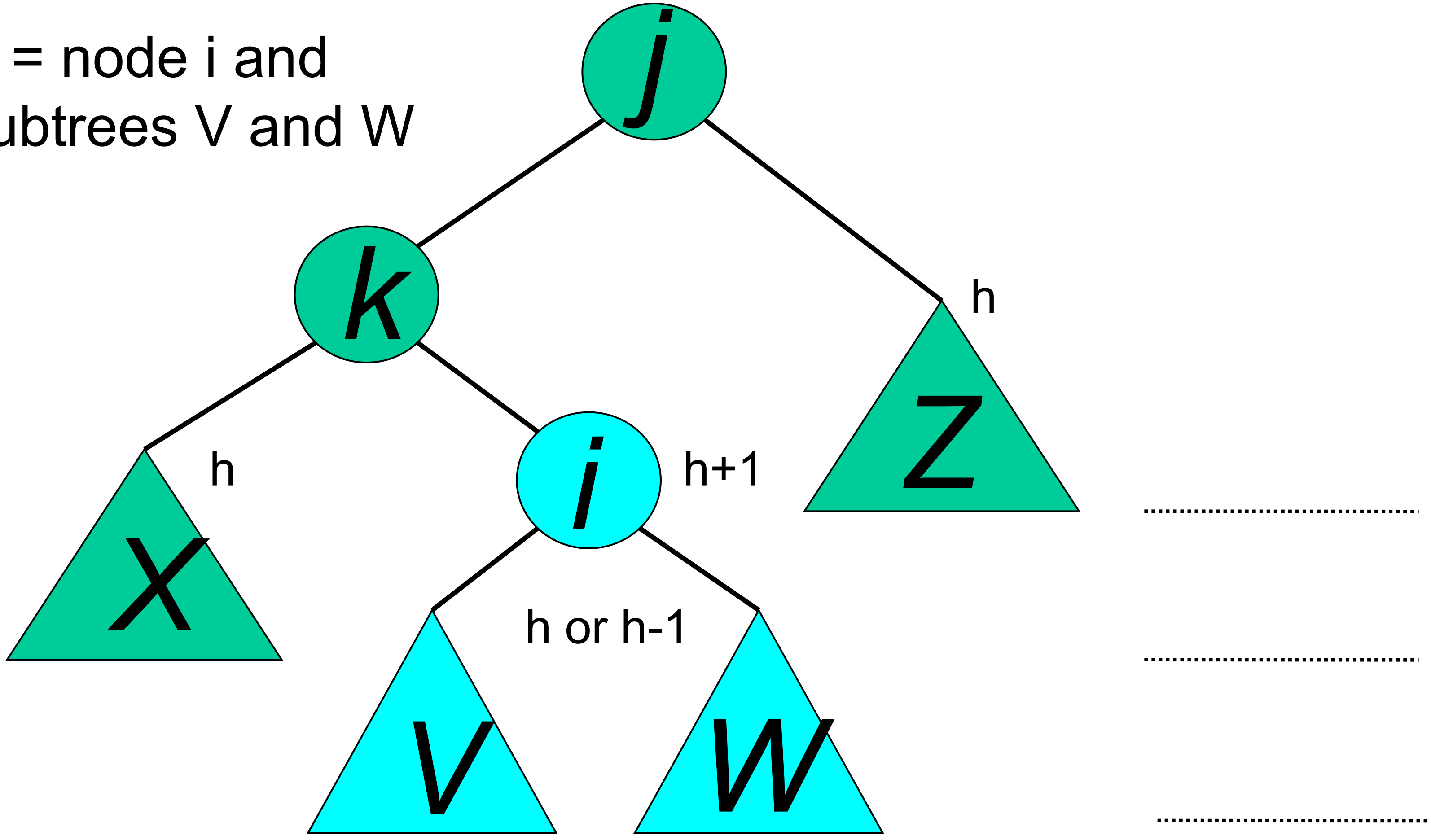
AVL Insertion: Inside Case

Consider the structure of subtree Y…

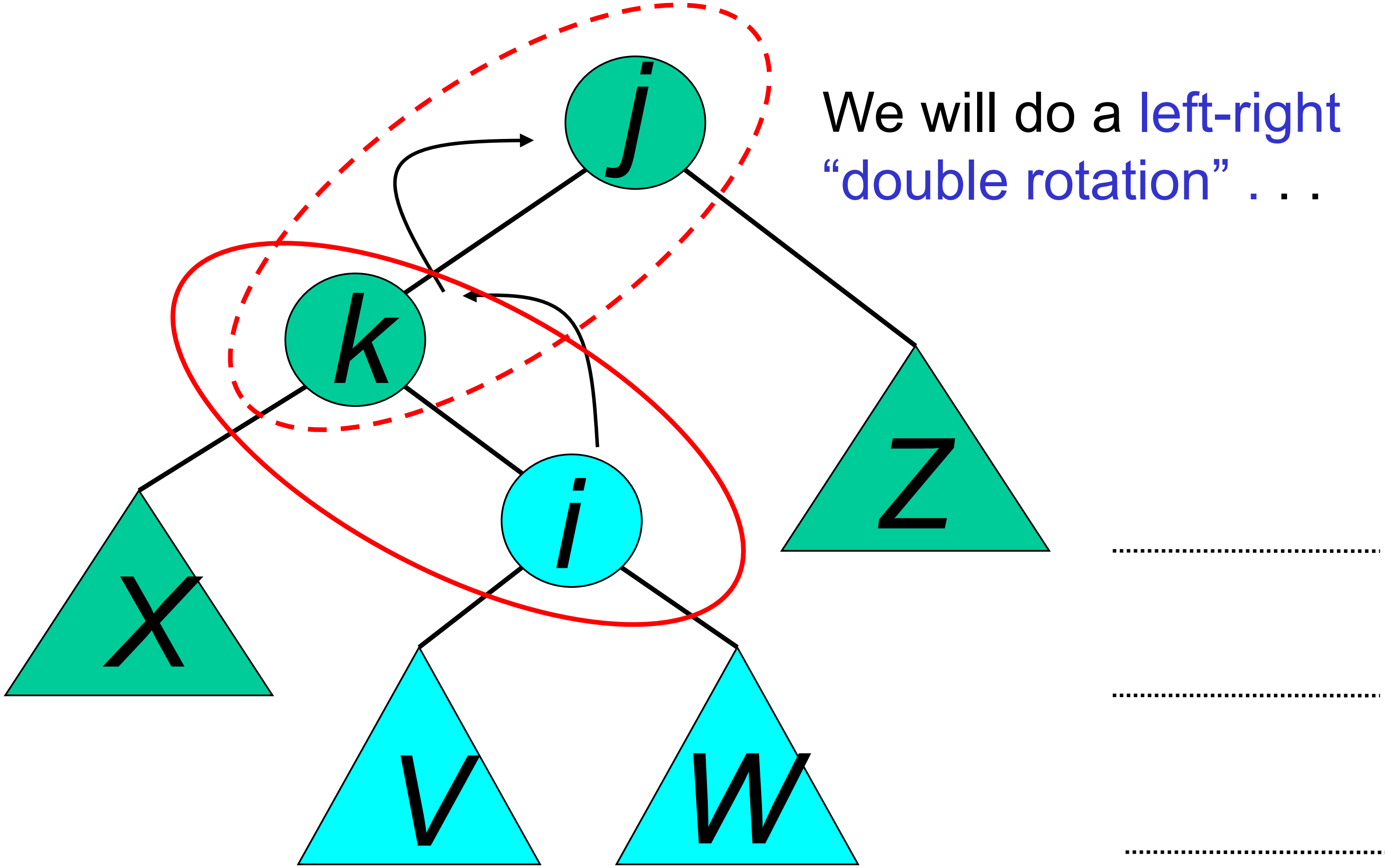


AVL Insertion: Inside Case

Y = node i and subtrees V and W

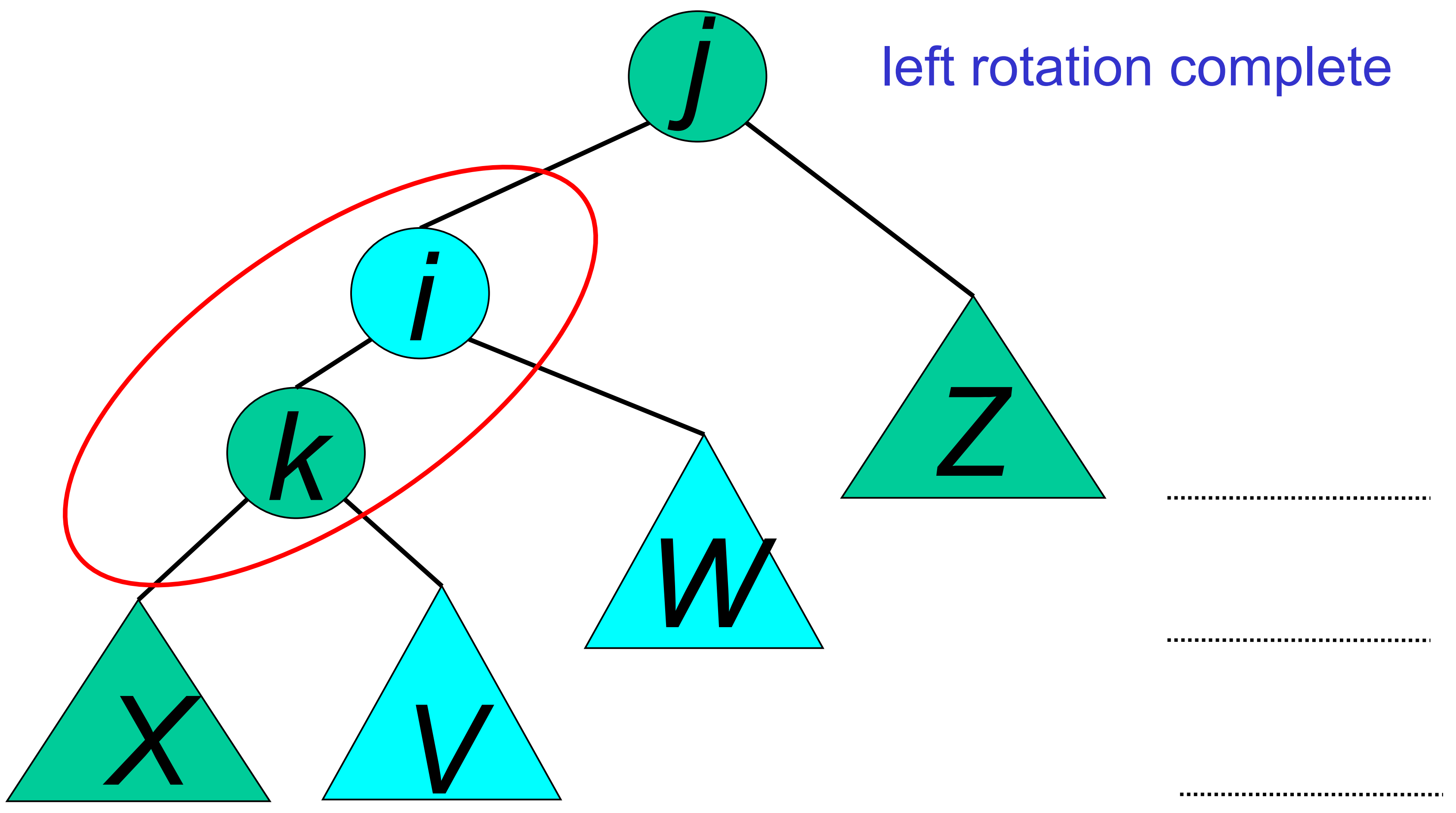


AVL Insertion: Inside Case

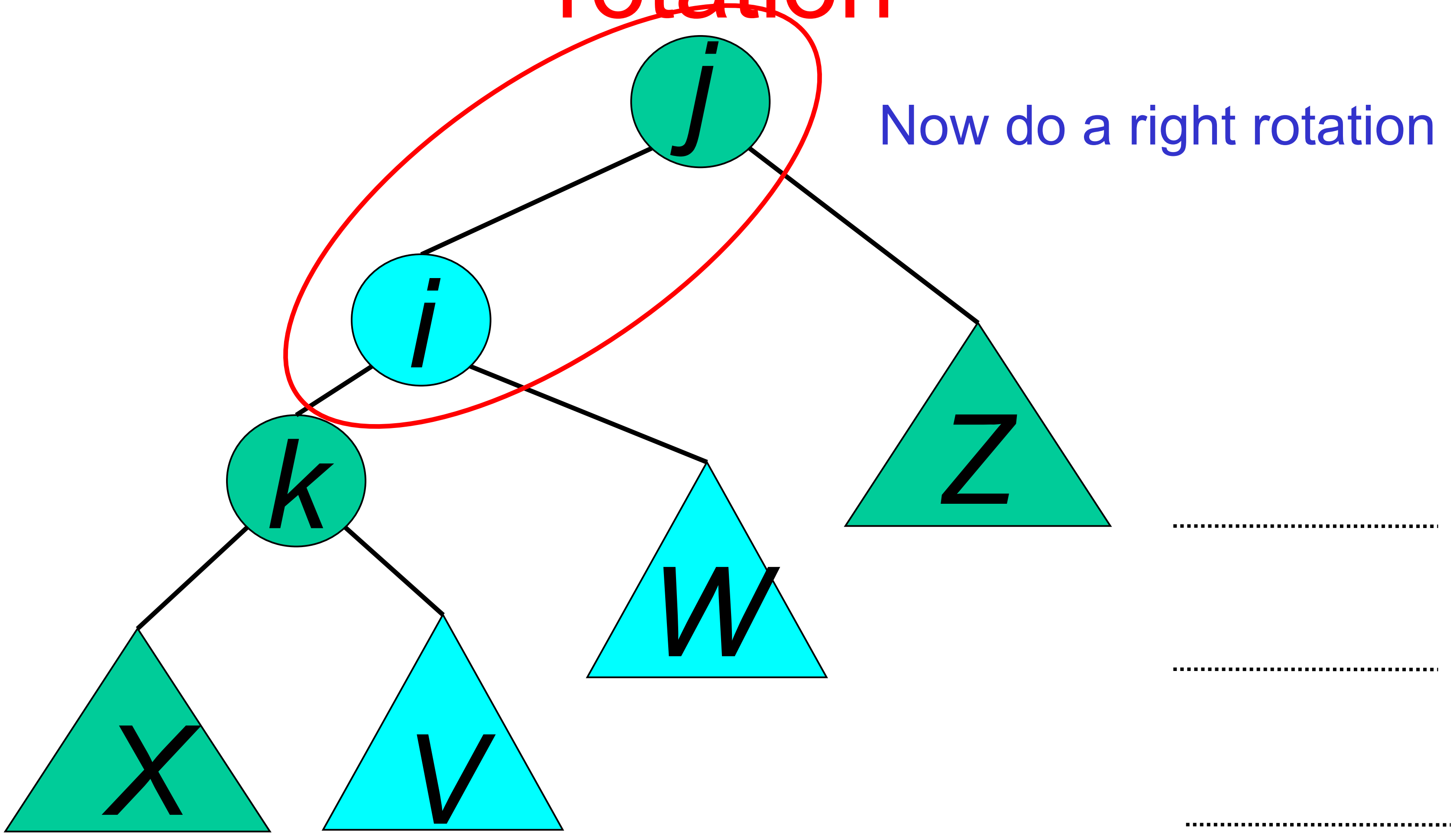


We will do a left-right "double rotation" . . .

Double rotation : first rotation



Double rotation : second rotation

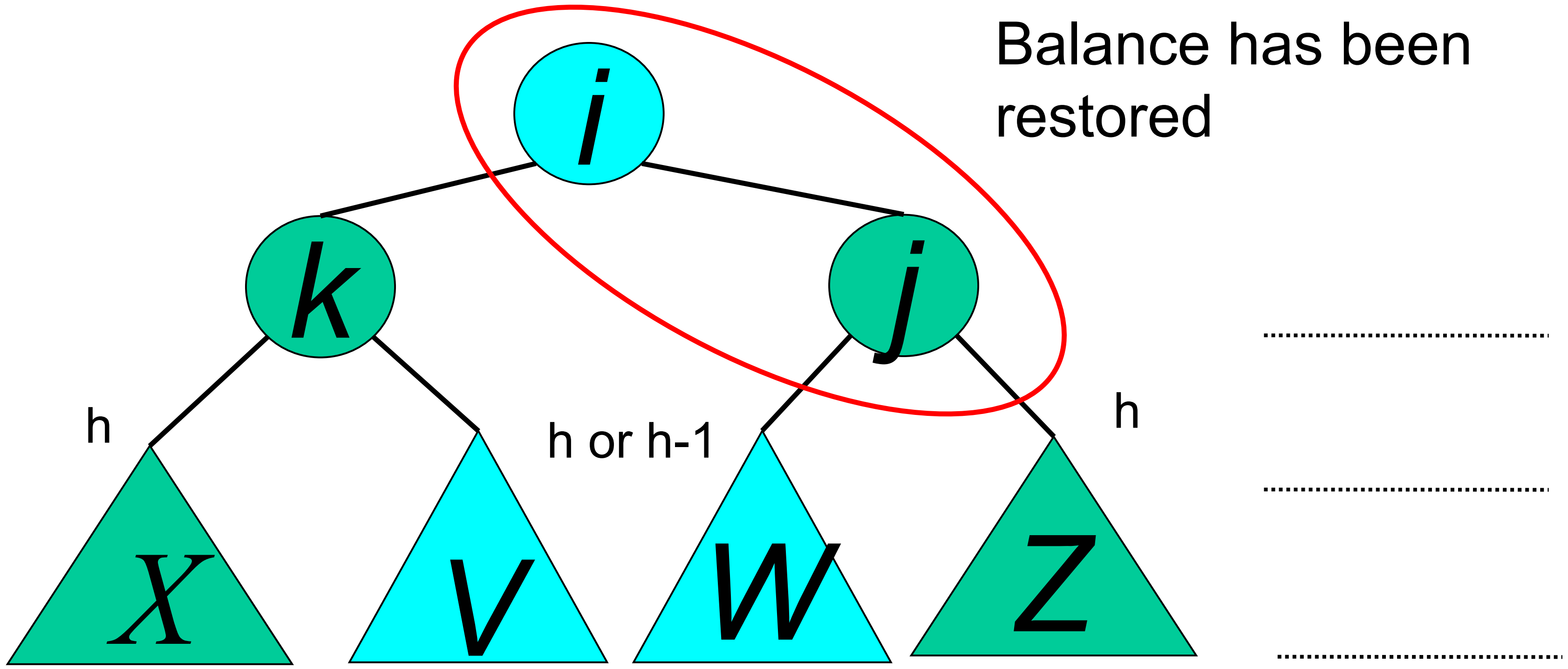


Now do a right rotation

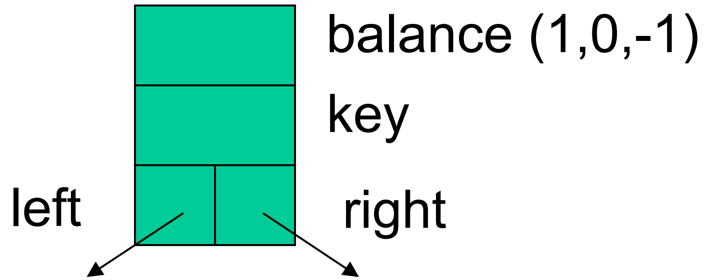
Double rotation : second rotation

right rotation complete

Balance has been restored



Implementation



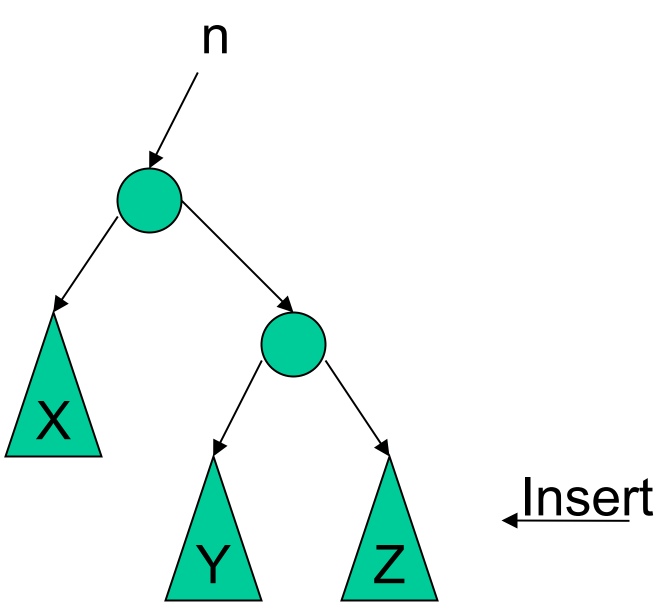
No need to keep the height; just the difference in height, i.e. the balance factor; this has to be modified on the path of insertion even if you don't perform rotations

Once you have performed a rotation (single or double) you won't need to go back up the tree

Single Rotation

```
RotateFromRight(n : reference node pointer) {
p : node pointer;
p := n.right;
n.right := p.left;
p.left := n;
n := p
}
```

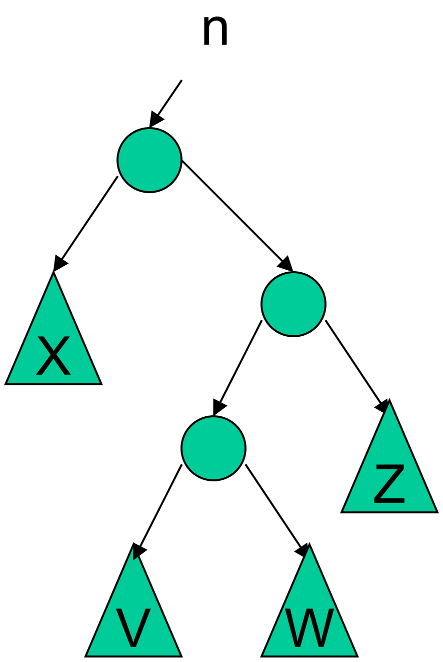
You also need to modify the heights or balance factors of n and p



Double Rotation

- Implement Double Rotation in two lines.

```
DoubleRotateFromRight(n : reference node pointer) {
????
}
```



Insertion in AVL Trees

- Insert at the leaf (as for all BST)
  - only nodes on the path from insertion point to root node have possibly changed in height
  - So after the Insert, go back up to the root node by node, updating heights
  - If a new balance factor (the difference h_{left}-h_{right}) is 2 or –2, adjust tree by *rotation* around the node

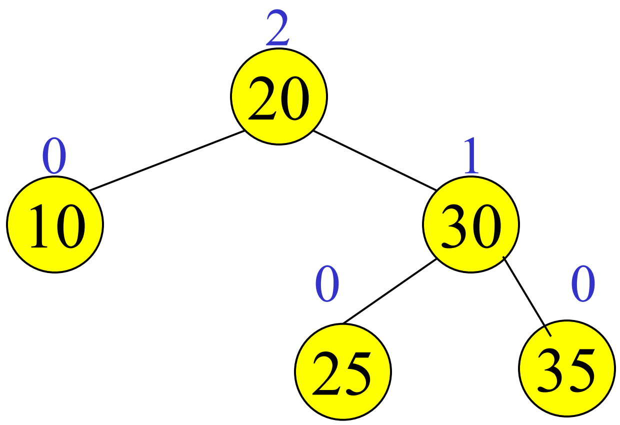
Insert in BST

```
Insert(T : reference tree pointer, x : element) : integer {
if T = null then
  T := new tree; T.data := x; return 1;//the links to
                                       //children are null
case
  T.data = x : return 0; //Duplicate do nothing
  T.data > x : return Insert(T.left, x);
  T.data < x : return Insert(T.right, x);
endcase
}
```

Insert in AVL trees

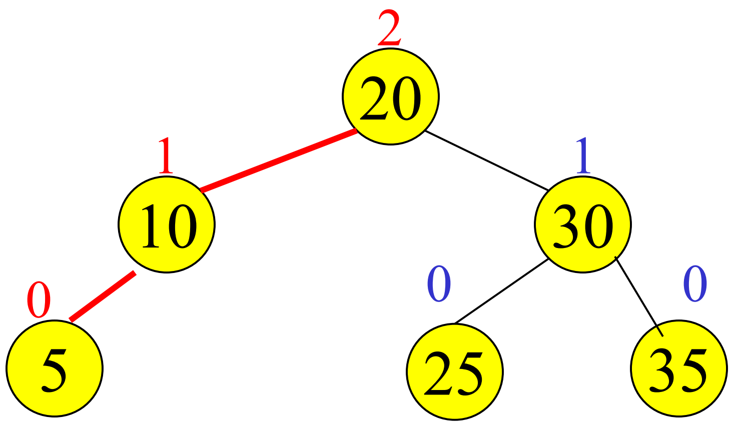
```
Insert(T : reference tree pointer, x : element) : {
if T = null then
  {T := new tree; T.data := x; height := 0; return;}
case
  T.data = x : return ; //Duplicate do nothing
  T.data > x : Insert(T.left, x);
               if ((height(T.left)- height(T.right)) = 2){
                  if (T.left.data > x ) then //outside case
                         T = RotatefromLeft (T);
                  else                       //inside case
                         T = DoubleRotatefromLeft (T);}
  T.data < x :  Insert(T.right, x);
                code similar to the left case
Endcase
  T.height := max(height(T.left),height(T.right)) +1;
  return;
}
```

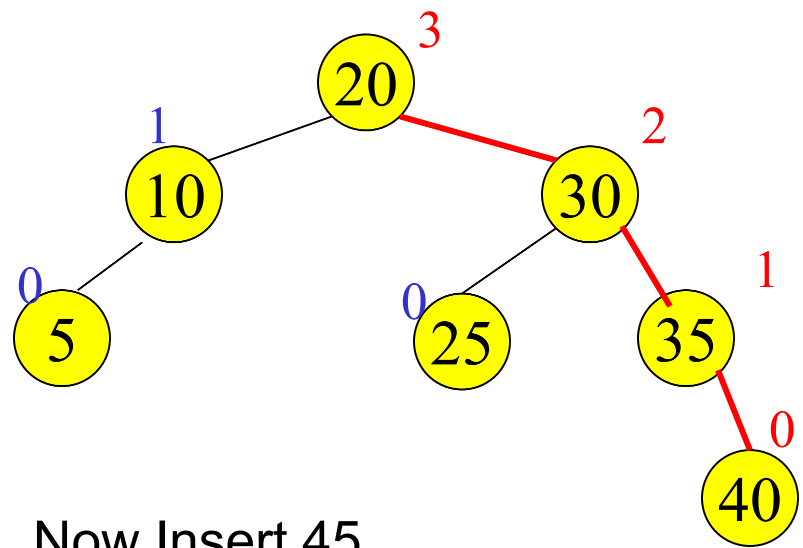
Example of Insertions in an AVL Tree



Insert 5, 40

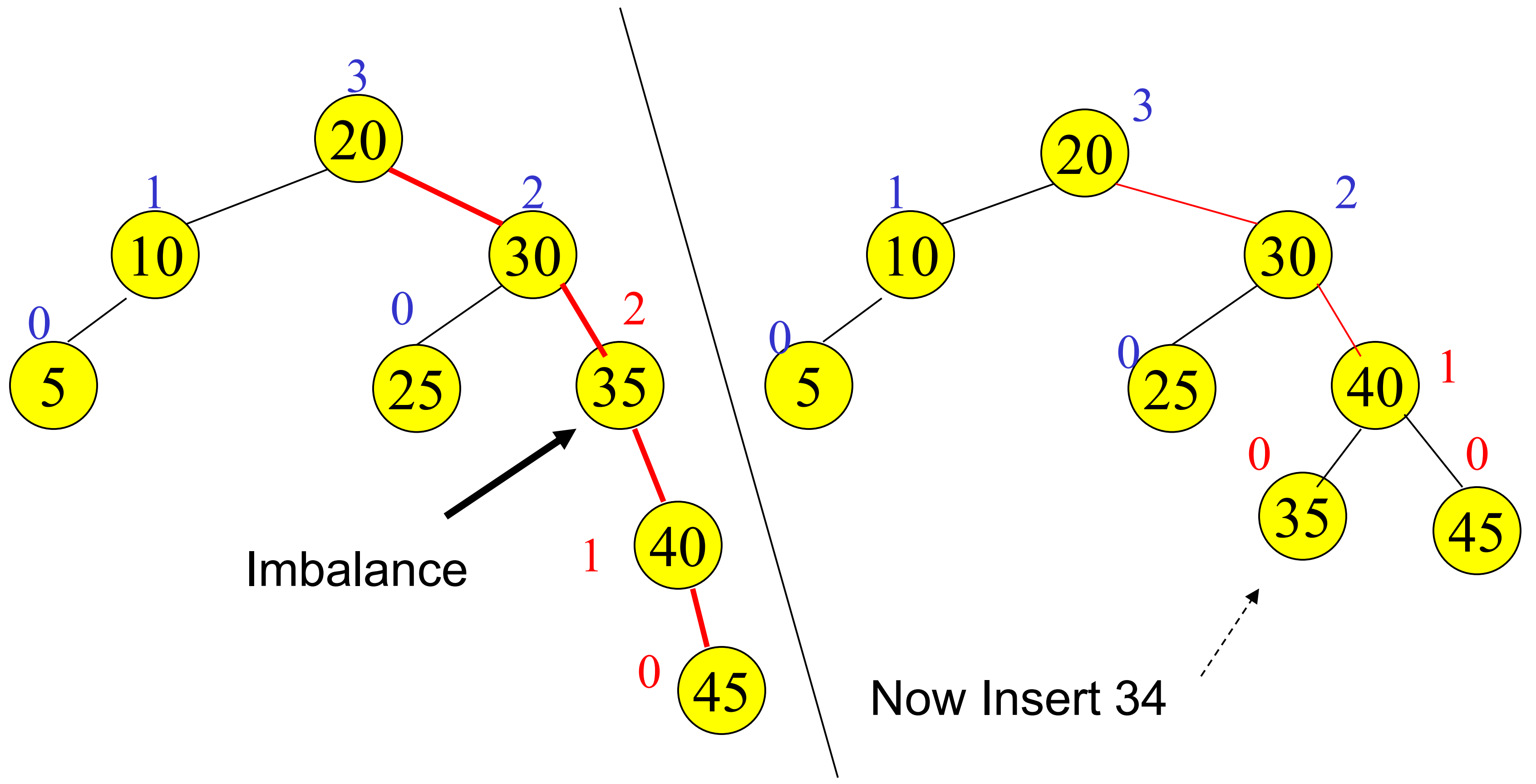
Example of Insertions in an AVL Tree



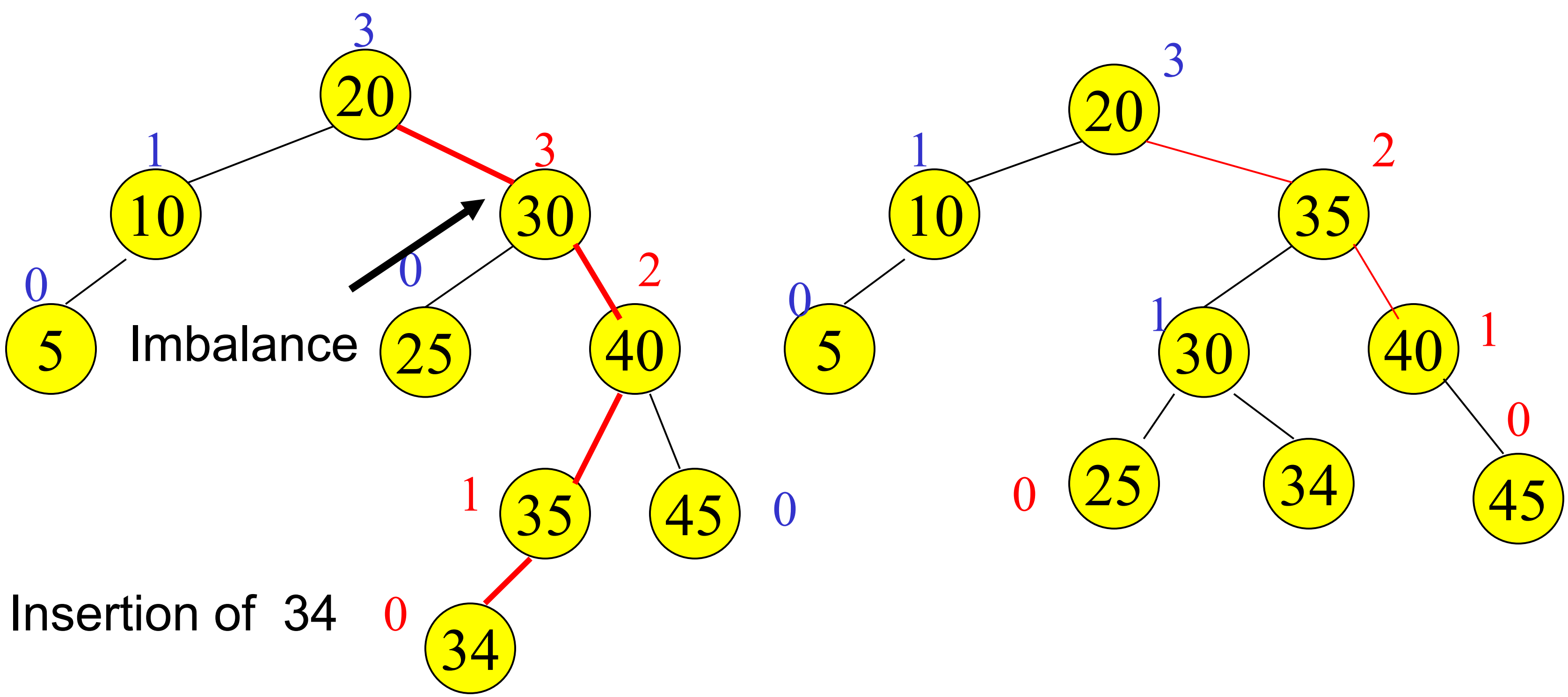


Now Insert 45

Single rotation (outside case)



Double rotation (inside case)



AVL Tree Deletion

- Similar but more complex than insertion
  - Rotations and double rotations needed to rebalance
  - Imbalance may propagate upward so that many rotations may be needed.

Pros and Cons of AVL Trees

Arguments for AVL trees:

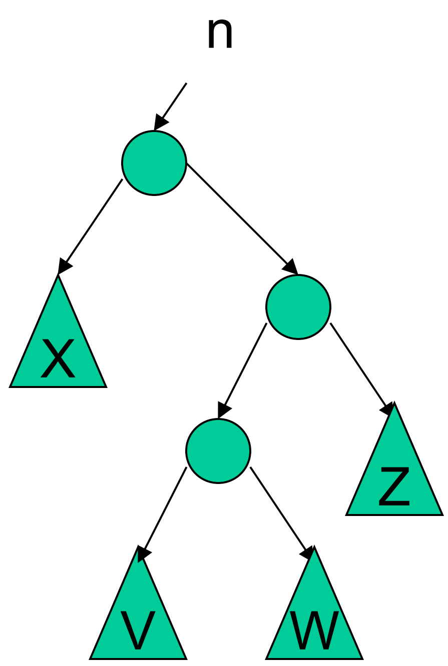
1. Search is O(log N) since AVL trees are always balanced.
2. Insertion and deletions are also O(logn)
3. The height balancing adds no more than a constant factor to the speed of insertion.

Arguments against using AVL trees:

1. Difficult to program & debug; more space for balance factor.
2. Asymptotically faster but rebalancing costs time.
3. Most large searches are done in database systems on disk and use other structures (e.g. B-trees).
4. May be OK to have O(N) for a single operation if total run time for many consecutive operations is fast (e.g. Splay trees).

Double Rotation Solution

```
DoubleRotateFromRight(n : reference node pointer) {
RotateFromLeft(n.right);
RotateFromRight(n);
}
```



Balanced search trees

Balanced search tree: A search-tree data structure for which a height of $O(\lg n)$ is guaranteed when implementing a dynamic set of n items.

Examples:

- AVL trees
- 2-3 trees
- 2-3-4 trees
- B-trees
- Red-black trees

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Red-black trees

This data structure requires an extra one-bit color field in each node.

Red-black properties:

1. Every node is either red or black.
2. The root and leaves (NIL's) are black.
3. If a node is red, then its parent is black.
4. All simple paths from any node x to a descendant leaf have the same number of black nodes = black-height(x).

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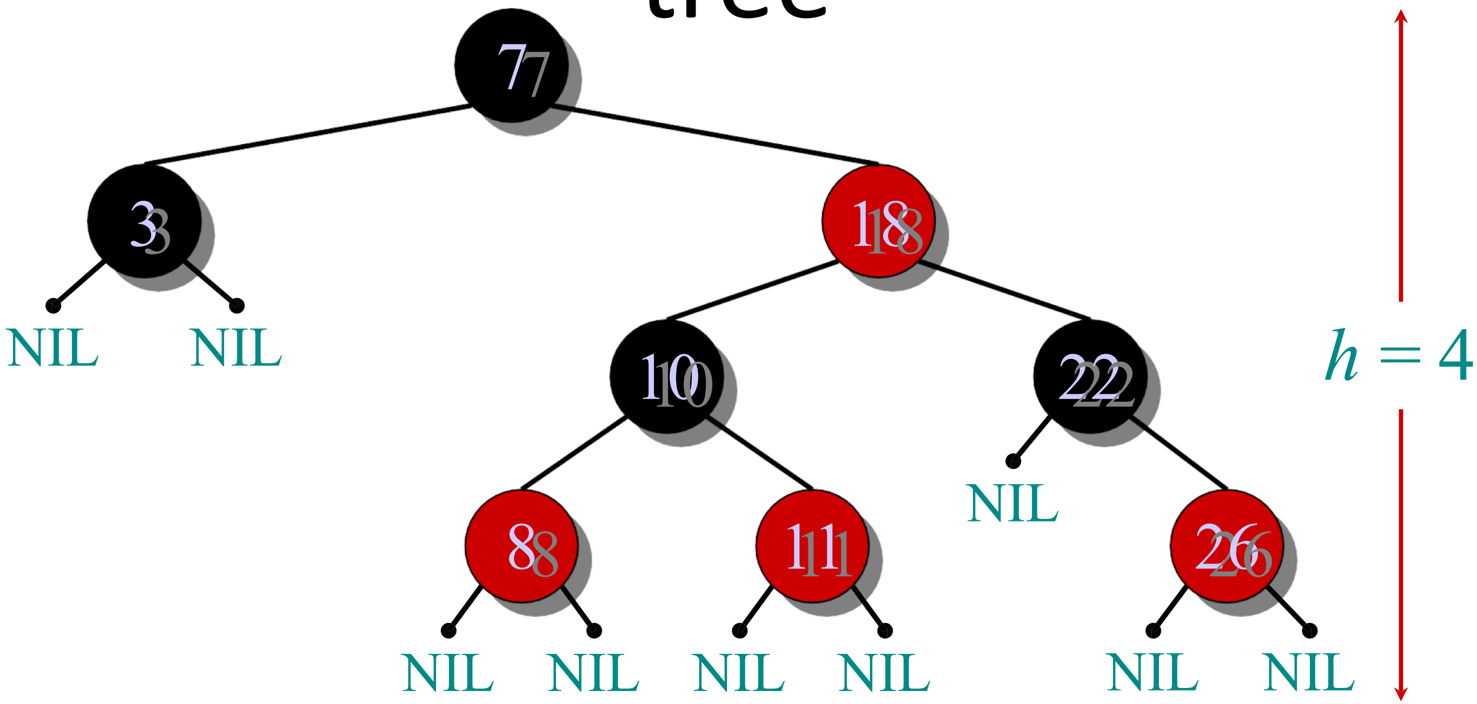
Example of a red-black tree



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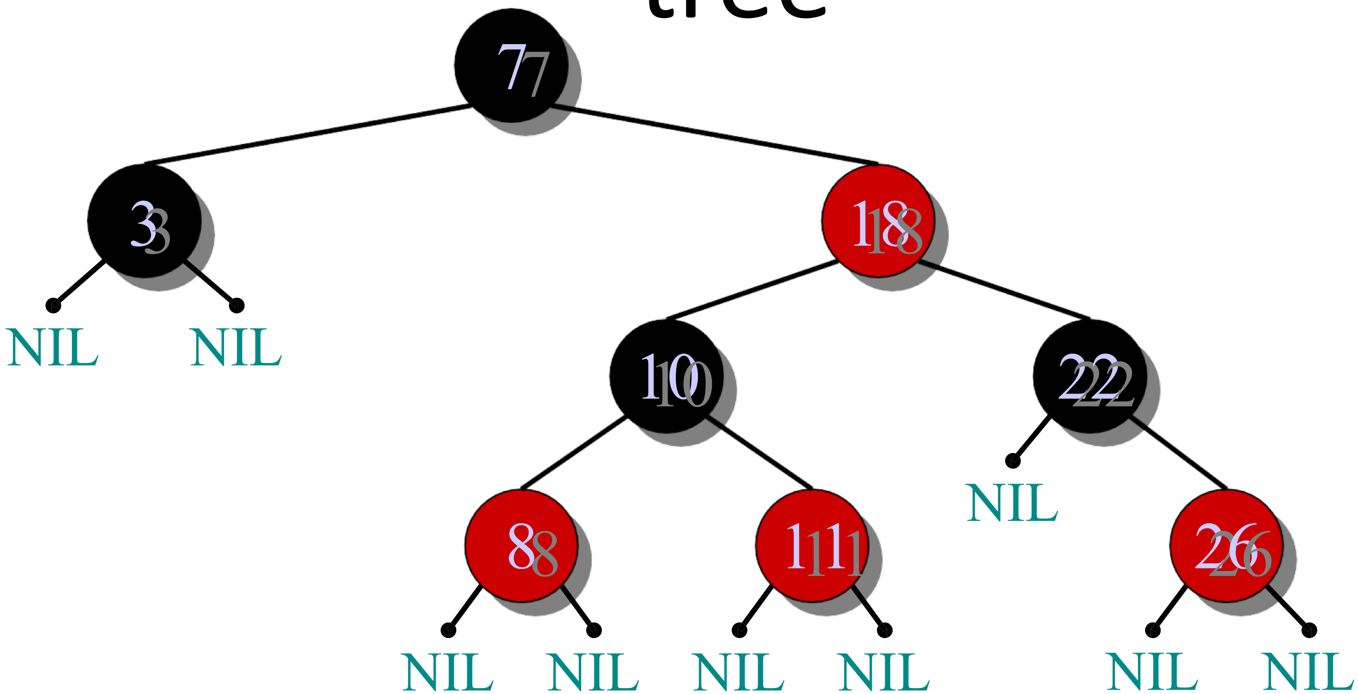

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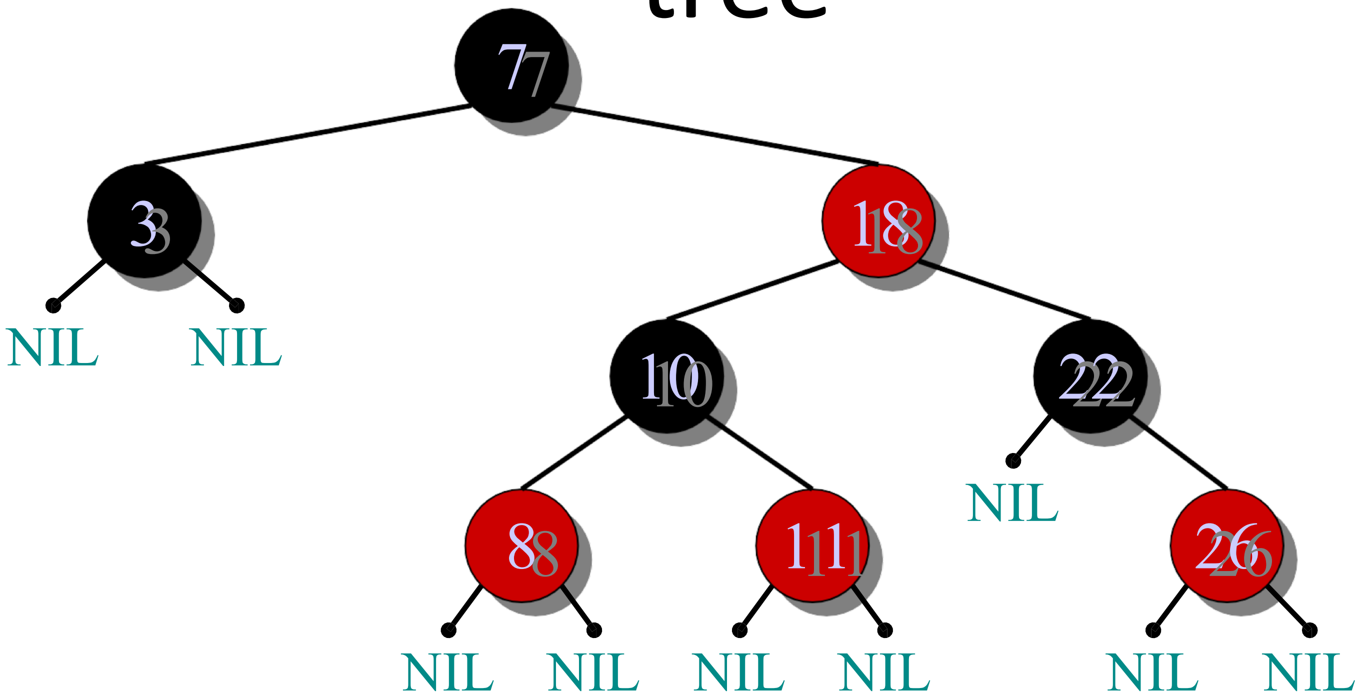

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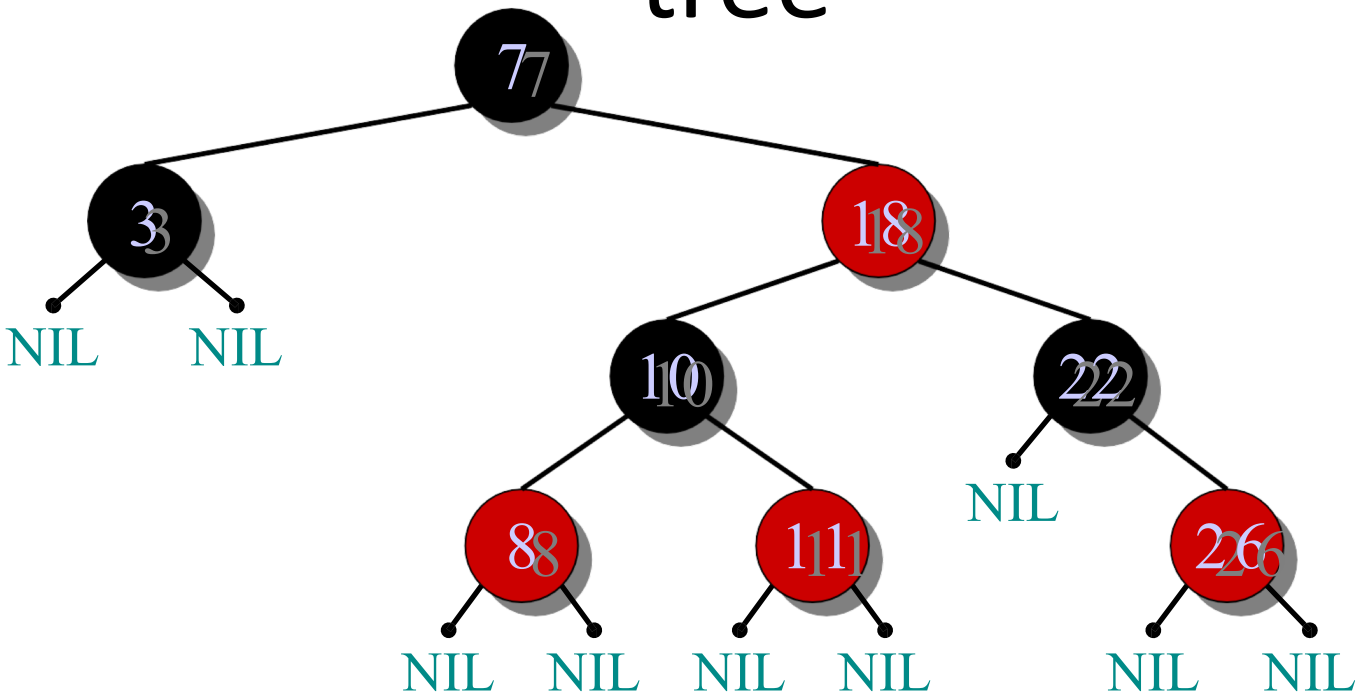

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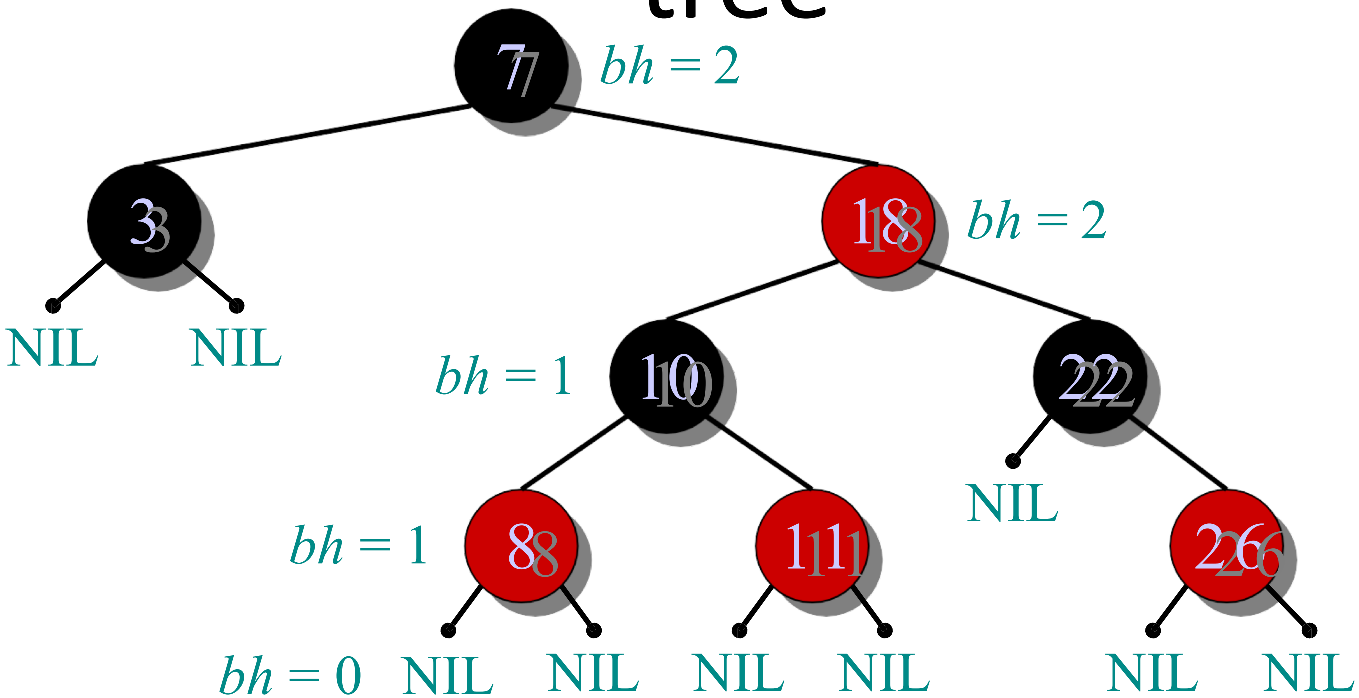

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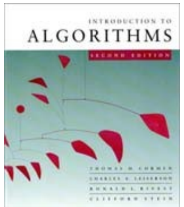


Height of a red-black tree

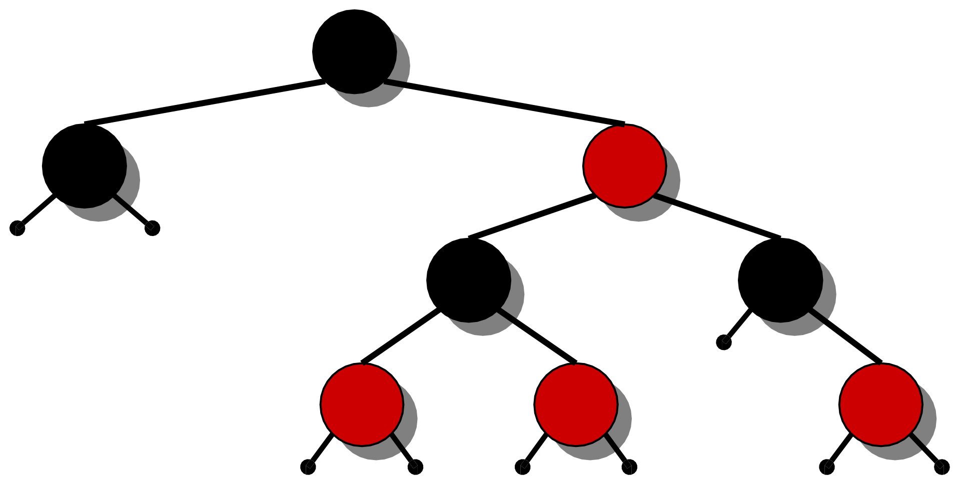
Theorem. A red-black tree with n keys has height

$$h \leq 2 \lg(n + 1).$$

Proof. (The book uses induction. Read carefully.)

INTUITION:

- Merge red nodes into their black parents.



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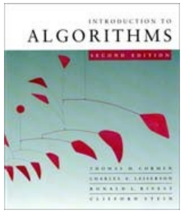

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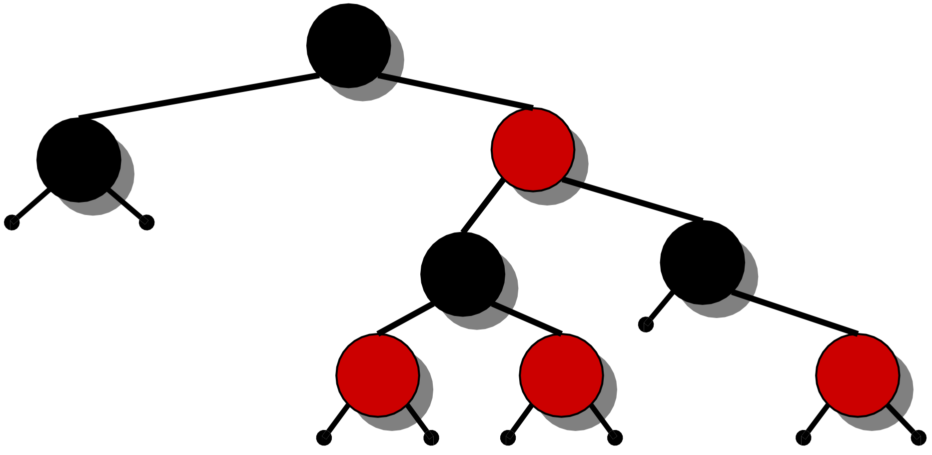

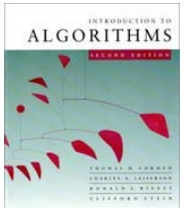

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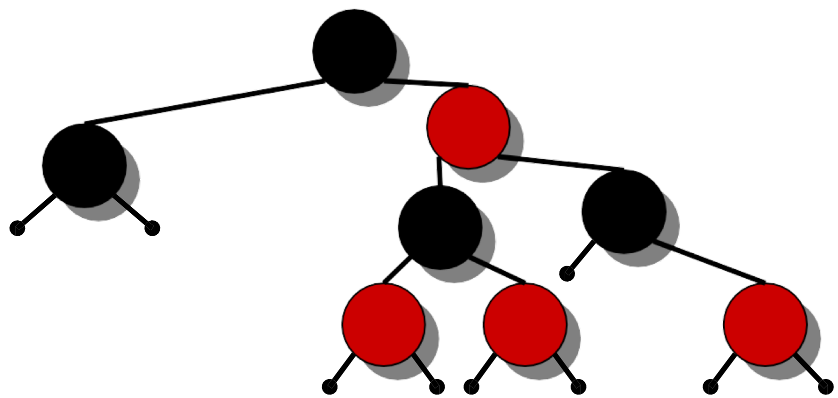

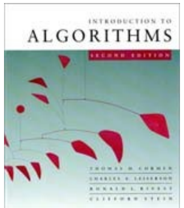

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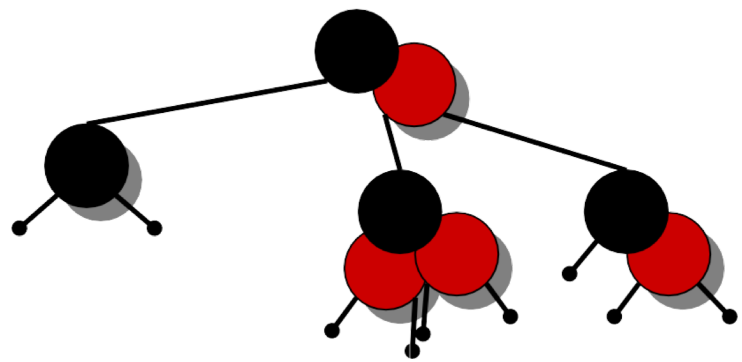

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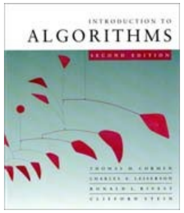

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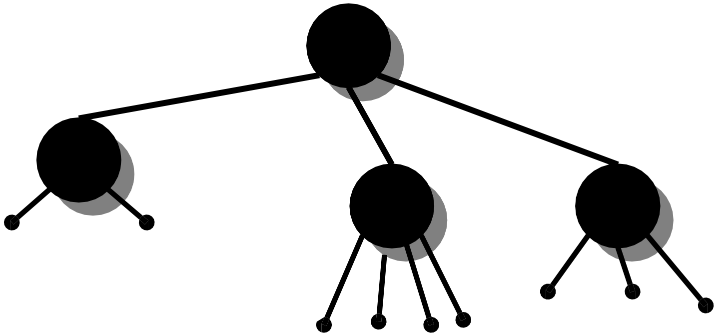

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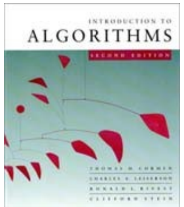

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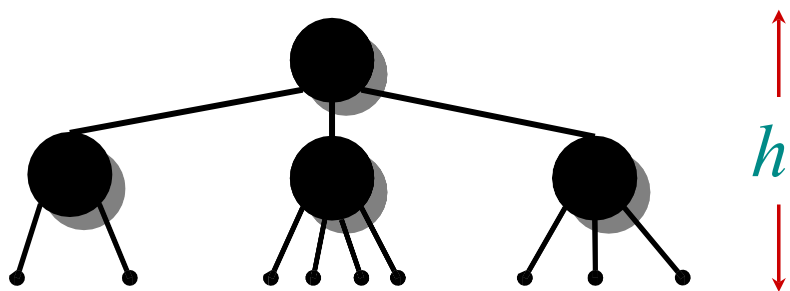
Proof. (The book uses induction. Read carefully.)

INTUITION:



- Merge red nodes into their black parents.
- This process produces a tree in which each node has 2, 3, or 4 children.
- The 2-3-4 tree has uniform depth h' of leaves.

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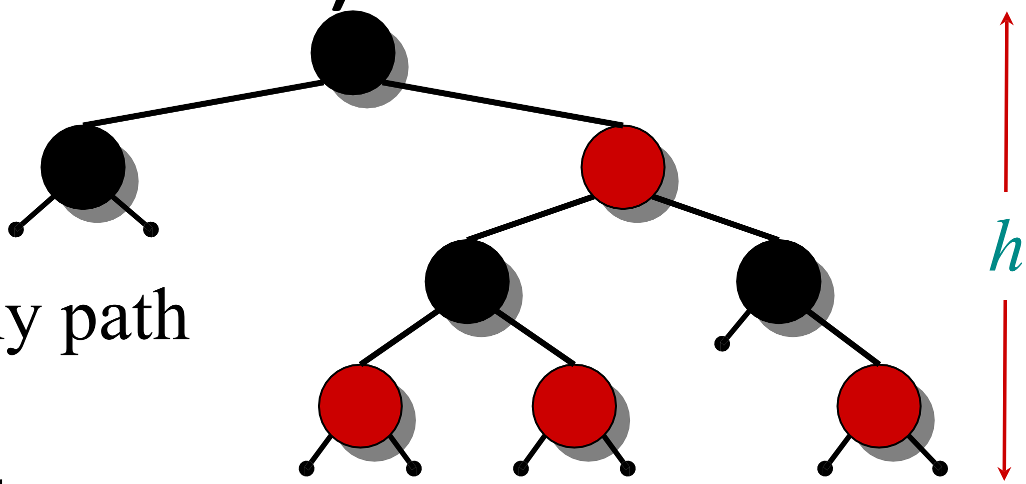
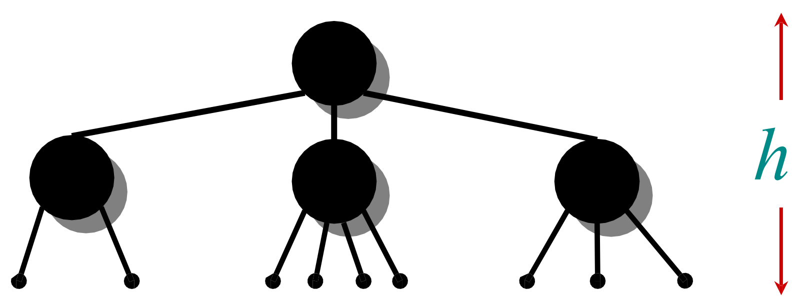
Proof (continued)

- We have $h' \geq h/2$, since at most half the leaves on any path are red.



- The number of leaves in each tree is $n + 1$
  $\Rightarrow n + 1 \geq 2^{h'}$
  $\Rightarrow \lg(n + 1) \geq h' \geq h/2$
  $\Rightarrow h \leq 2 \lg(n + 1)$. $\square$



Query operations

Corollary. The queries SEARCH, MIN, MAX, SUCCESSOR, and PREDECESSOR all run in $O(\lg n)$ time on a red-black tree with n nodes.

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Modifying operations

The operations INSERT and DELETE cause modifications to the red-black tree:

- the operation itself,
- color changes,
- restructuring the links of the tree via ***"rotations"***.

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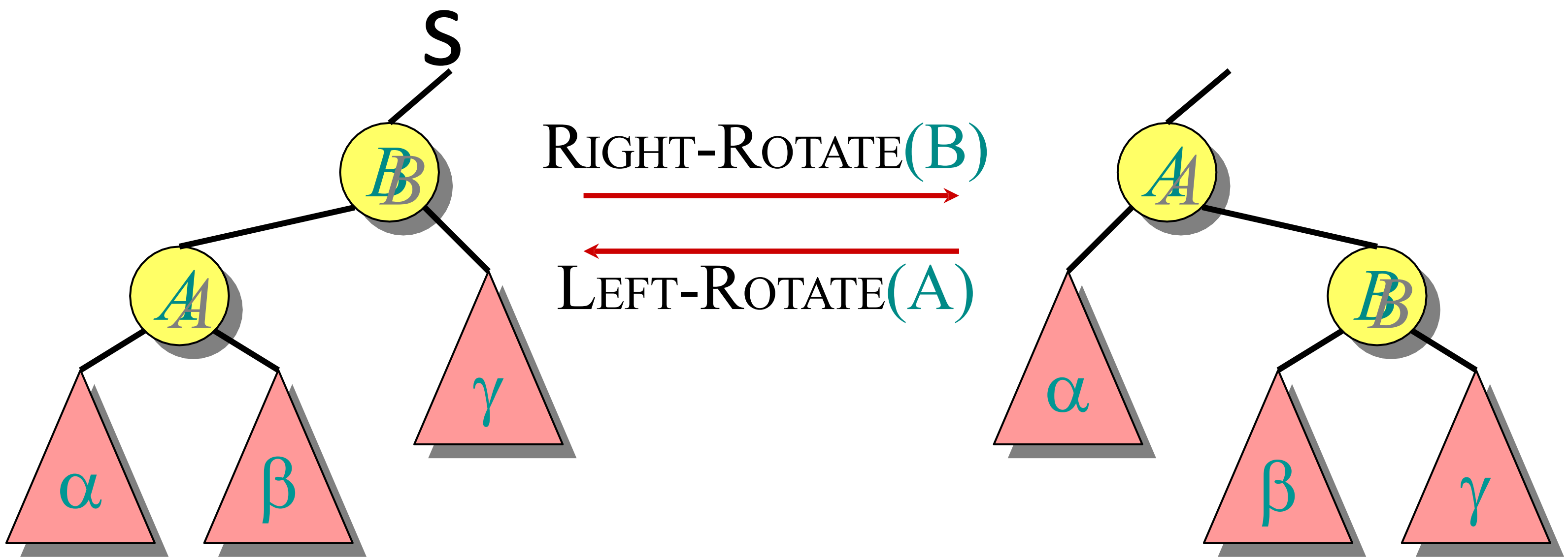
Rotation



Rotations maintain the inorder ordering of keys:

- $a \in \alpha, b \in \beta, c \in \gamma \;\Rightarrow\; a \le A \le b \le B \le c$.

A rotation can be performed in $O(1)$ time.

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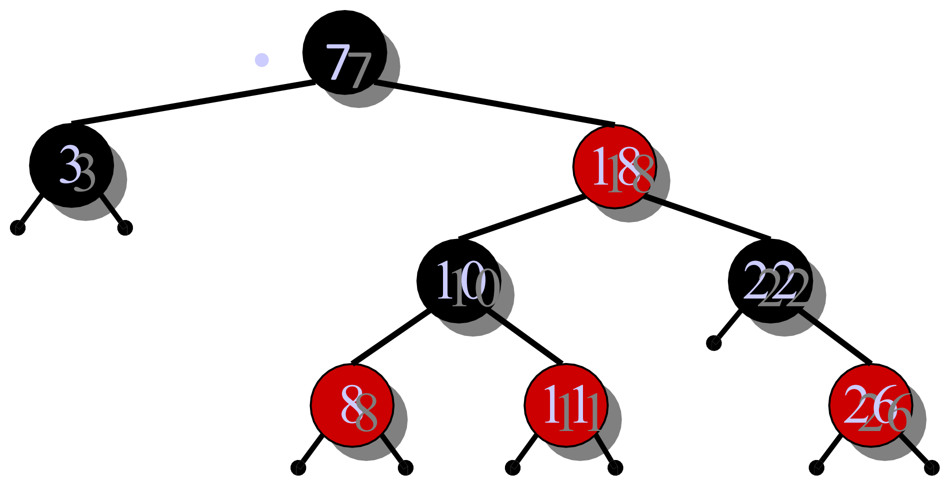
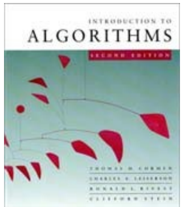
Insertion into a red-black tree

- **IDEA:** Insert *x* in tree. Color *x* red. Only red-black property 3 might be violated. Move the violation up the tree by recoloring until it can be fixed with rotations and recoloring.

- **Example:**



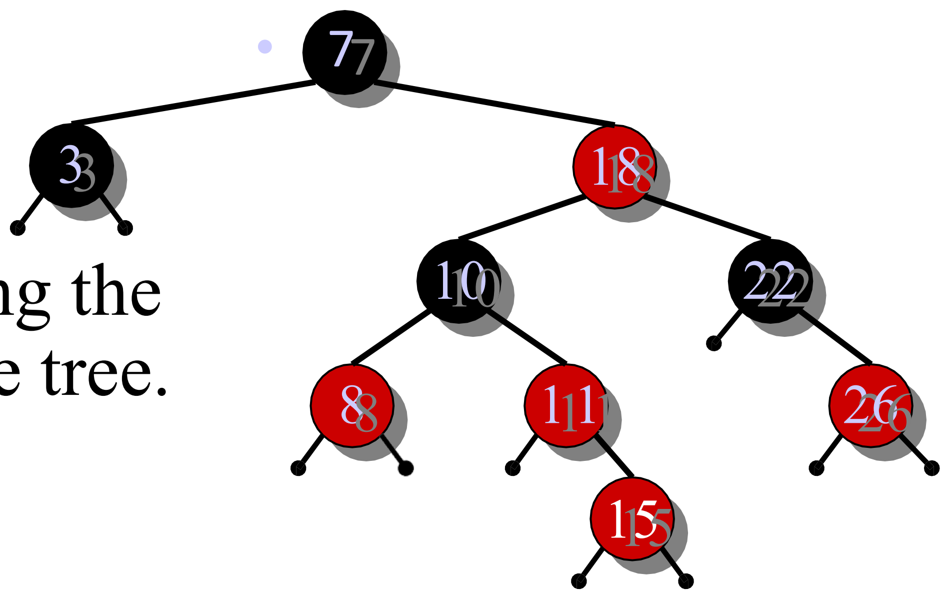


Insertion into a red-black tree

•**IDEA:** Insert x in tree. Color x red. Only red- black property 3 might be violated. Move the violation up the tree by recoloring until it can be fixed with rotations and recoloring.

Example:

- Insert $x = 15$.
- Recolor, moving the violation up the tree.



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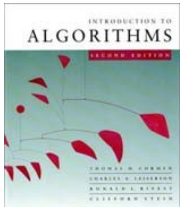

Insertion into a red-black tree

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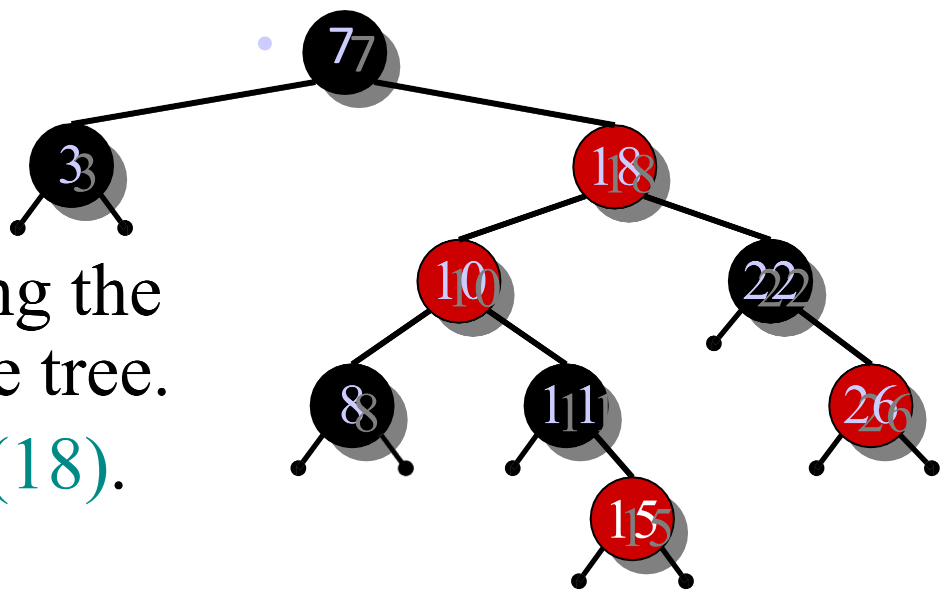
- Insert $x = 15$.
- Recolor, moving the violation up the tree.
- RIGHT-ROTATE(18).



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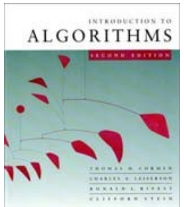
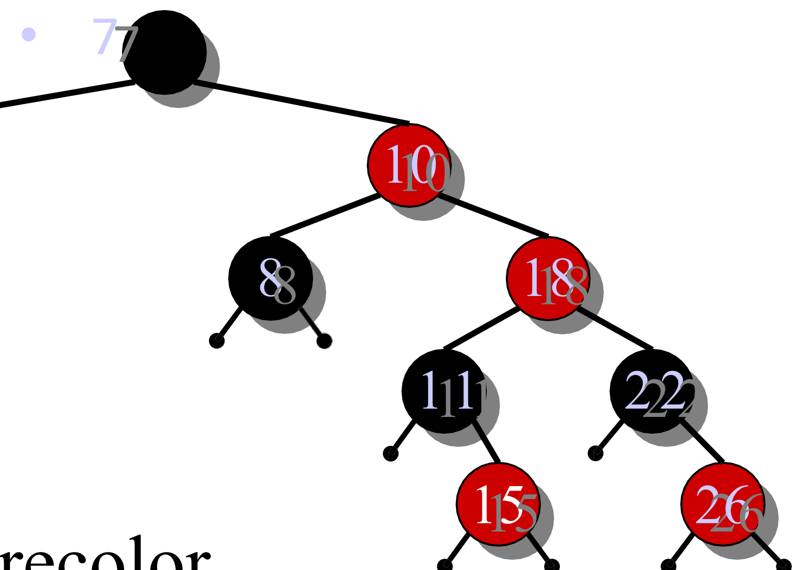


Insertion into a red-black tree

•**IDEA:** Insert *x* in tree. Color *x* red. Only red- black property 3 might be violated. Move the violation up the tree by recoloring until it can be fixed with rotations and recoloring.



Example:

- Insert $x = 15$.
- Recolor, moving the violation up the tree.
- RIGHT-ROTATE(18).
- LEFT-ROTATE(7) and recolor.

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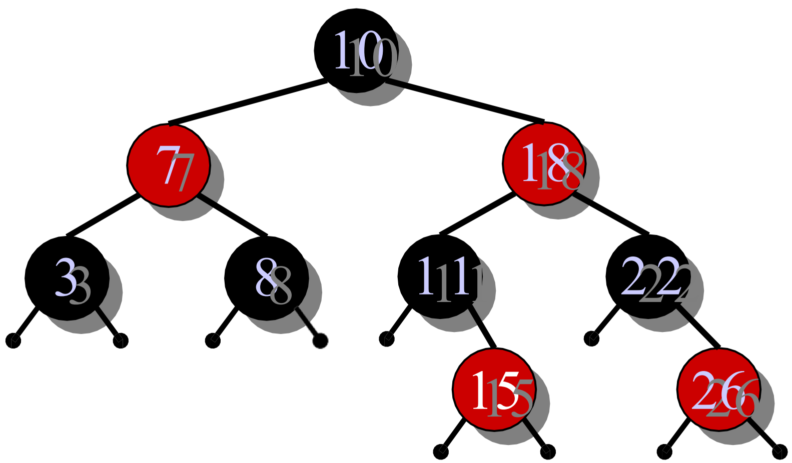
Insertion into a red-black tree

IDEA: Insert x in tree. Color x red. Only red-black property 3 might be violated. Move the violation up the tree by recoloring until it can be fixed with rotations and recoloring.

Example:

- Insert $x = 15$.
- Recolor, moving the violation up the tree.
- RIGHT-ROTATE(18).
- LEFT-ROTATE(7) and recolor.



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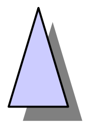
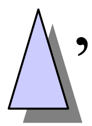
Pseudocode

RB-INSERT(T, x)
TREE-INSERT(T, x)
$color[x] \leftarrow$ RED ◁only RB property 3 can be violated
while $x \neq root[T]$ and $color[p[x]] =$ RED
do if $p[x] = left[p[p[x]]]$
then $y \leftarrow right[p[p[x]]]$ ◁y = aunt/uncle of x
if $color[y] =$ RED
then ⟨**Case 1**⟩
else if $x = right[p[x]]$
then ⟨**Case 2**⟩ ◁Case 2 falls into Case 3
⟨**Case 3**⟩
else ⟨**"then"** clause with "*left*" and "*right*" swapped⟩
$color[root[T]] \leftarrow$ BLACK

Graphical notation

Let  denote a subtree with a black root.

All ’s have the same black-height.

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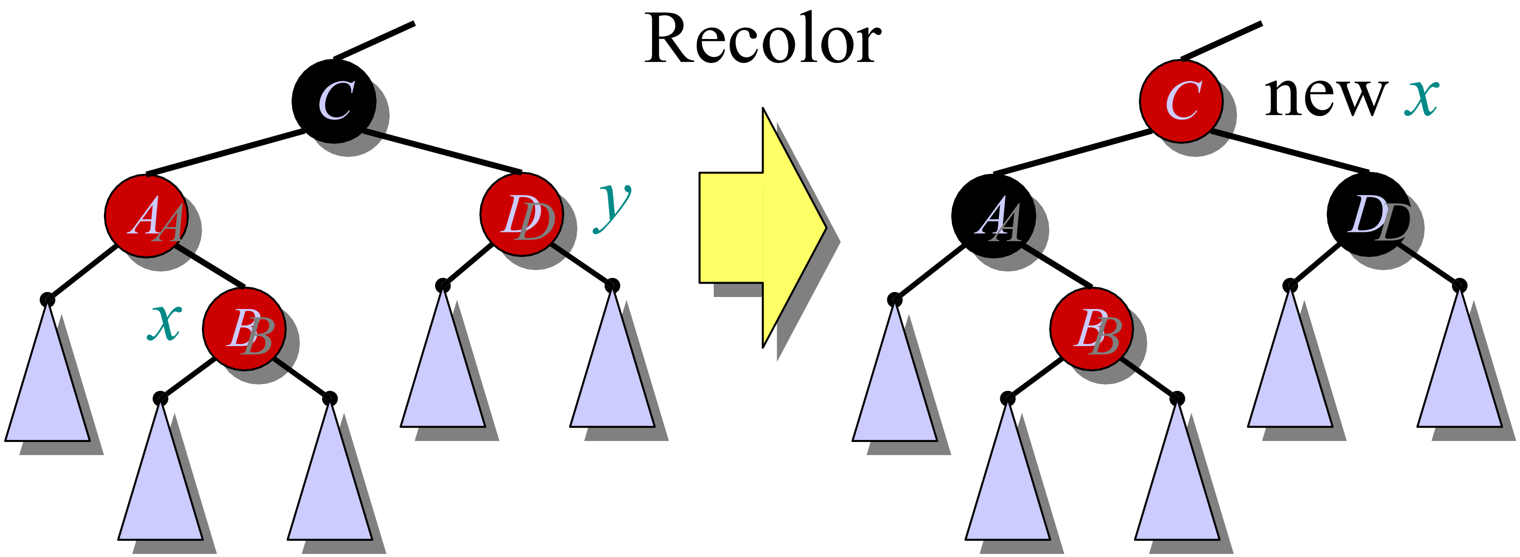
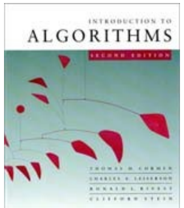
Case 1

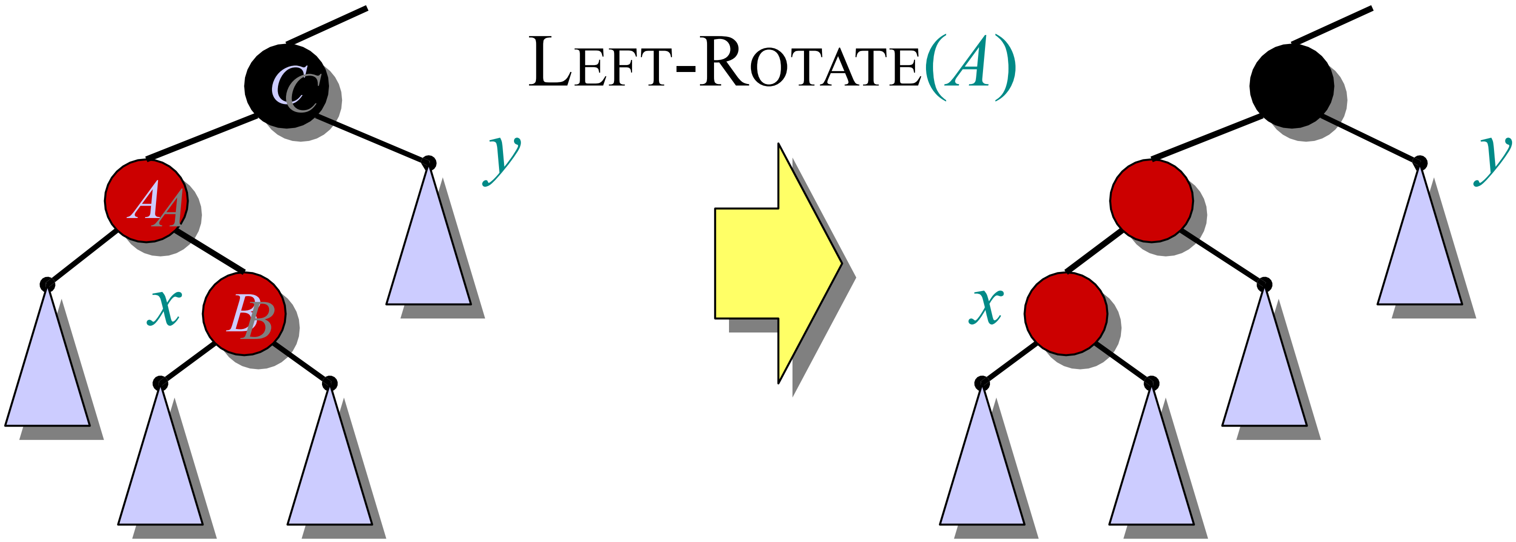


(Or, children of *A* are swapped.)

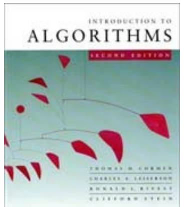
Push *C*'s black onto *A* and *D*, and recurse, since *C*'s parent may be red.

Case 2

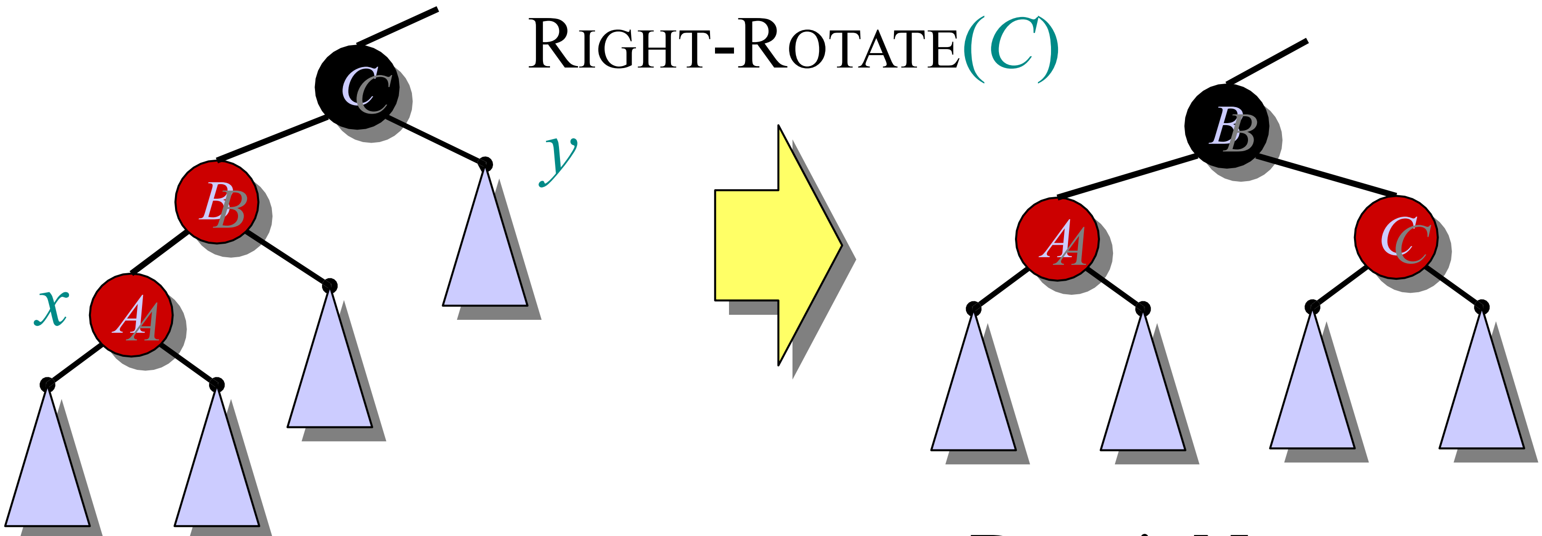




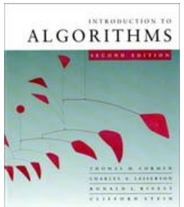
Transform to Case 3.



Case 3



Done! No more violations of RB property 3 are possible.



Analysis

- Go up the tree performing Case 1, which only recolors nodes.
- If Case 2 or Case 3 occurs, perform 1 or 2 rotations, and terminate.

Running time: $O(\lg n)$ with $O(1)$ rotations.

RB-DELETE — same asymptotic running time and number of rotations as RB-INSERT (see textbook).

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