Shortest Path Algorithms

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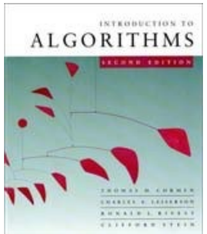


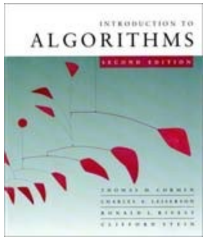
Single-source shortest paths

Problem. From a given source vertex $s \in V$, find the shortest-path weights $\delta(s, v)$ for all $v \in V$.

If all edge weights $w(u, v)$ are *nonnegative*, all shortest-path weights must exist.

Idea: Greedy.

1. Maintain a set S of vertices whose shortest-path distances from s are known.
2. At each step add to S the vertex $v \in V - S$ whose distance estimate from s is minimal.
3. Update the distance estimates of vertices adjacent to v.



Dijkstra's algorithm

$d[s] \leftarrow 0$

for each $v \in V - \{s\}$

    do $d[v] \leftarrow \infty$

$S \leftarrow \varnothing$

$Q \leftarrow V$ $\rhd$ Q is a priority queue maintaining $V - S$

while $Q \neq \varnothing$

    do $u \leftarrow$ EXTRACT-MIN(Q)

        $S \leftarrow S \cup \{u\}$

        for each $v \in Adj[u]$

            do if $d[v] > d[u] + w(u, v)$

                then $d[v] \leftarrow d[u] + w(u, v)$

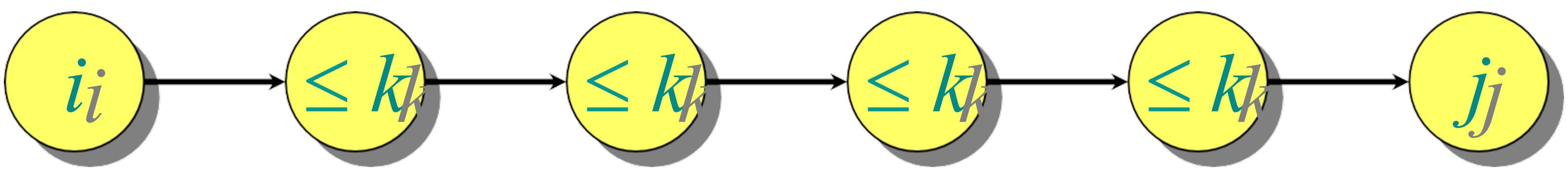
All-pairs shortest paths

Input: Digraph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, with edge-weight function $w : E \to \mathbb{R}$.
Output: $n \times n$ matrix of shortest-path lengths $\delta(i, j)$ for all $i, j \in V$.

Floyd-Warshall algorithm

Also dynamic programming, but faster!

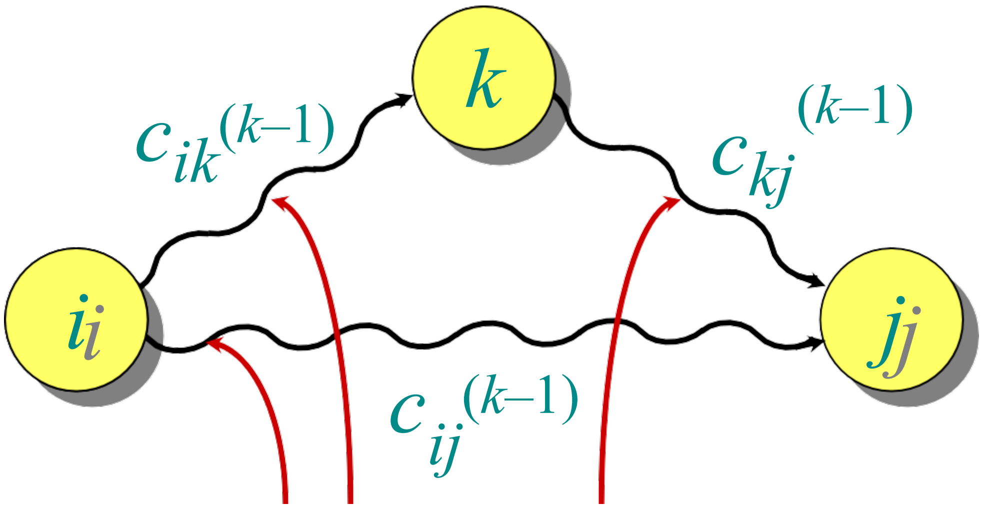
Define $c_{ij}^{(k)} =$ weight of a shortest path from i to j with intermediate vertices belonging to the set $\{1, 2, \ldots, k\}$.



Thus, $\delta(i, j) = c_{ij}^{(n)}$. Also, $c_{ij}^{(0)} = a_{ij}$.

Floyd-Warshall recurrence

$$c_{ij}^{(k)} = \min_k \{c_{ij}^{(k-1)}, c_{ik}^{(k-1)} + c_{kj}^{(k-1)}\}$$



intermediate vertices in $\{1, 2, \ldots, k\}$

Pseudocode for Floyd-Warshall

for $k \leftarrow 1$ **to** n
  do for $i \leftarrow 1$ **to** n
    do for $j \leftarrow 1$ **to** n
      do if $c_{ij} > c_{ik} + c_{kj}$
        then $c_{ij} \leftarrow c_{ik} + c_{kj}$

relaxation

Notes:

- Okay to omit superscripts, since extra relaxations can't hurt.
- Runs in $\Theta(n^3)$ time.
- Simple to code.
- Efficient in practice.