Programming Intelligent Physical Systems Examination

IIT Kharagpur, Tuesday, 13 August, 2019

[Open Book Exam, Calculator Allowed, Computer NOT Allowed]
[<u>Time allowed: 60 mins</u>] [Maximum Marks: 200]

Question 1 [20 marks]

Consider a dynamical system with the following system equation:

$$\ddot{x} + 3\dot{x} + 2x = 5u.$$

The input and output of the system is respectively u and $y = x$. The initial conditions are $x(0) = 0$, $\dot{x}(0) = 0$. Is the system stable? Give reasons for your answer.

Solution 1

Using the Laplace transform,

$s^2\mathbf{X}(s) - sx(0) - \dot{x}(0) + 3(s\mathbf{X}(s) - x(0)) + 2\mathbf{X}(s) = 5\mathbf{U}(s)$ **[5 marks]**

If $x(0) = 0$, $\dot{x}(0) = 0$, the system equation in frequency domain is simplified to

$$(s^2 + 3s + 2)\mathbf{X}(s) = 5\mathbf{U}(s)$$

$$G(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \frac{\mathbf{X}(s)}{\mathbf{U}(s)} = \frac{5}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)}$$

Correct $G(s)$ **[5 marks]**

So the system has two poles $p_1 = -1$, $p_2 = -2$ and no zeros.

Correct system poles [5 marks]

All system poles negative $\rightarrow$ the system is stable.

System is stable [5 marks]

Question 2 [20 marks]

Consider a dynamical system with the following state-space model

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

Design the feedback and feedforward gains such that the closed-loop system has poles $p_1 = -1$, $p_2 = -1$.

Solution 2

Let the control law be: $u = \mathbf{K}\mathbf{x} + Fr$.

Close-loop system:

$$\dot{\mathbf{x}} = A\mathbf{x} + B(\mathbf{K}\mathbf{x} + Fr) = (A + B\mathbf{K})\mathbf{x} + BFr$$

$$A_{cl} = A + B\mathbf{K} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ k_1 - 6 & k_2 + 5 \end{bmatrix}$$

Correct A_{cl} [5 marks]

Characteristic equation of the close-loop system:

$$det(\lambda\mathbf{I} - A_{cl}) = 0$$

$$det(\lambda\mathbf{I} - A_{cl}) = det(\begin{bmatrix} \lambda & -1 \\ 6 - k_1 & \lambda - k_2 - 5 \end{bmatrix}) = \lambda^2 - (k_2 + 5)\lambda + 6 - k_1$$

Correct characteristic equation A_{cl} [5 marks]

System has poles $p_1 = -1$, $p_2 = -1$, this implies that the characteristic equation: $\lambda^2 + 2\lambda + 1 = 0$. Hence, $-(k_2 + 5) = 2$, $6 - k_1 = 1$, and therefore, $k_1 = 5$, $k_2 = -7$.

Correct k_1 and k_2, i.e., the feedback gain K [5 marks]

$$\begin{aligned} F &= \frac{1}{C(-A - B\mathbf{K})^{-1}B} \\ &= \frac{1}{\begin{bmatrix} 1 & 0 \end{bmatrix} (\begin{bmatrix} 0 & -1 \\ 6 & -5 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 5 & -7 \end{bmatrix})^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}} \\ &= \frac{1}{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}} \\ &= 1 \end{aligned}$$

Hence, $\mathbf{K} = \begin{bmatrix} 5 & -7 \end{bmatrix}$ and $F = 1$. **Correct feedforward gain F [5 marks]**

Question 3 [40 marks]

Consider the following task set running on a single processor.

T_i	p_i[ms]	D_i[ms]	$e_i[ms]$
T_1	10	5	3
T_2	15	8	3
T_3	20	10	5

What is the processor's utilization? What is the demand bound function for this processor over the interval $[0, 30]$ms? What is the processor's load in this interval? Is the task set schedulable?

Solution 3

Utilization $U = \sum_{i=1}^{n} \frac{e_i}{p_i} = \frac{e_1}{p_1} + \frac{e_2}{p_2} + \frac{e_3}{p_3} = \frac{3}{10} + \frac{3}{15} + \frac{5}{20} = 0.75$.

Correct utilization [5 marks]

The demand bound function: $h(t) = \sum_{i=1}^{n} max(0, \lfloor \frac{t-D_i}{p_i} + 1 \rfloor)e_i$.
$\forall i, D_i < p_i \rightarrow h(t) = \lfloor \frac{t-D_1}{p_1} + 1 \rfloor e_1 + \lfloor \frac{t-D_2}{p_2} + 1 \rfloor e_2 + \lfloor \frac{t-D_3}{p_3} + 1 \rfloor e_3 = \lfloor \frac{t-5}{10} + 1 \rfloor 3 + \lfloor \frac{t-8}{15} + 1 \rfloor 3 + \lfloor \frac{t-10}{20} + 1 \rfloor 5$.

Events at: $t = 5, 8, 10, 15, 23, 25, 30$.

Identifying the correct times at which the demand bound function changes [10 marks]

$$h(0) = 0, h(5) = 3, h(8) = 6, h(10) = 11, h(13) = 14, h(23) = 17, h(25) = 20, h(30) = 25.$$

All correct values of the function $h()$ [10 marks]. Each incorrect h results in -2 marks. Minimum possible marks is 0.

Processor load in $[0, 30]$ms: load $= max(\frac{h(t)}{t}) = \frac{h(10)}{10} = 1.1 > 1.0$.

Identifying $t = 10$ [5 marks]. Correct value of load at $t = 10$ [5 marks].

Hence, the task set is not schedulable.

Not schedulable [5 marks]

Question 4

Consider a control plant with the following continuous-time dynamics:

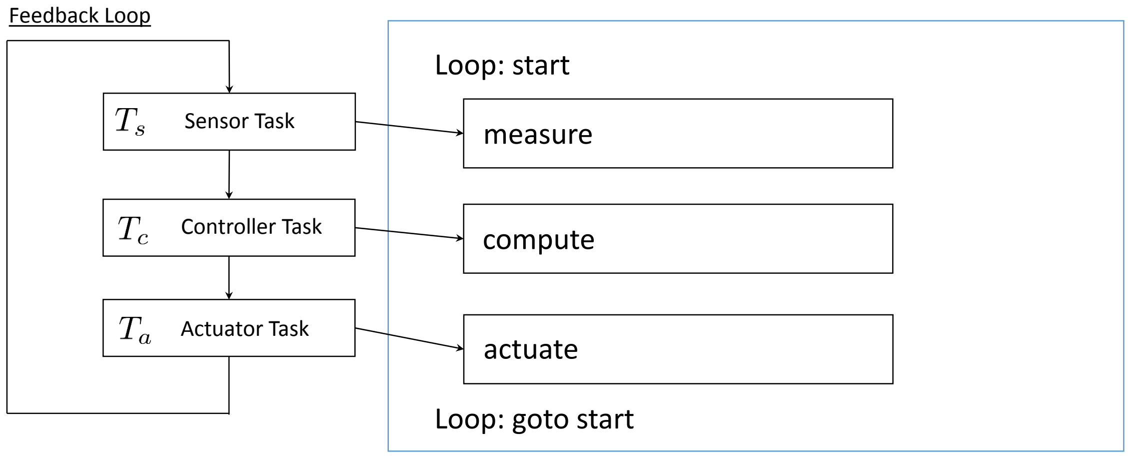
$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Assume that the above system is sampled using a ZOH with constant sampling period h and the sensor-to-actuator delay is represented with τ, where $\tau \leq h$.

The control objective for the system is to achieve $y[k] \rightarrow 0$ as $k \rightarrow \infty$ from any initial condition $x[0]$. Towards implementing a controller to achieve the above objective, the control software is implemented onto a processor. The processor is running a real-time operating system which follows a fixed priority preemptive scheduler. Moreover, the processor also runs three other periodic hard real-time tasks which are represented as follows,

Tasks	p_i[ms]	D_i[ms]	$e_i[ms]$
T_1	5	5	1
T_2	25	25	4
T_3	15	12	2

where p_i, D_i and e_i are respectively period, relative deadline and WCET of task T_i. The control software for the system is partitioned in the following way: three tasks T_a (actuator task), T_c (compute task) and T_s (sensor task).



- Clearly, implementation of a controller needs to map three additional tasks T_a, T_c and T_s onto the processor.
- T_s and T_a have WCET times 0.2ms and T_c has WCET 2ms.
- For a given sampling period h of the controller, $T_a : \{p_i, D_i, e_i\} = \{h, 0.05h, 0.2ms\}$, $T_s : \{p_i, D_i, e_i\} = \{h, 0.05h, 0.2ms\}$, $T_c : \{p_i, D_i, e_i\} = \{h, 0.4h, 2ms\}$.
- Because of thermal constraints, the maximum utilization U_{max} of the processor can be no more than 60%.
- In order to facilitate easier implementation of the current and also future applications, all the tasks running on the processor should have periods that are multiples of 5ms.

Question 4(a) [45 marks]
What is the minimum sampling period h such that all real-time and control tasks are schedulable? The scheduling scheme to be used on the processor is the deadline monotonic scheduling. Assign priorities to the tasks and perform a response time analysis and show the results. Is the system schedulable?

Solution 4(a)

Processor utilization:

$$U = \frac{1}{5} + \frac{4}{25} + \frac{2}{15} + \frac{0.2}{h} + \frac{0.2}{h} + \frac{2}{h} \leq 0.6$$

Hence, since $h \geq 22.5ms$, let us assign $h = 25ms$.

Correct sampling period h [10 marks]

Scheduling according to DM:

Tasks	p_i[ms]	D_i[ms]	$e_i[ms]$	$prio$
T_a	25	1.25	0.2	1
T_s	25	1.25	0.2	2
T_1	5	5	1	3
T_c	25	10	2	4
T_3	15	12	2	5
T_2	25	25	4	6

Response time analysis:

$R_a^0 = 0, R_a^1 = 0.2, R_a^2 = 0.2, R_a = 0.2 < 1.25 \rightarrow$ deadline met
$R_s^0 = 0, R_s^1 = 0.2, R_s^2 = 0.4, R_s^3 = 0.4, R_s = 0.4 < 1.25 \rightarrow$ deadline met
$R_1^0 = 0, R_1^1 = 1, R_1^2 = 1.4, R_1^3 = 1.4, R_1 = 1.4 < 5 \rightarrow$ deadline met
$R_c^0 = 0, R_c^1 = 2, R_c^2 = 3.4, R_c^3 = 3.4, R_c = 3.4 < 10 \rightarrow$ deadline met
$R_3^0 = 0, R_3^1 = 2, R_3^2 = 5.4, R_3^3 = 6.4, R_3^4 = 6.4, R_3 = 6.4 < 12 \rightarrow$ deadline met
$R_2^0 = 0, R_2^1 = 4, R_2^2 = 9.4, R_2^3 = 10.4, R_2^4 = 11.4, R_2^5 = 11.4, R_2 = 11.4 < 25 \rightarrow$ deadline met

Hence, the task set is schedulable with the chosen sampling period.

Correct response times of all tasks including R_a, R_s and R_c [30 marks, 5 marks for each correct answer]

All tasks are schedulable [5 marks]

Question 4(b) [75 marks]
Now consider the continuous-time dynamics described below.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

With the previous choice of sampling period h, design a state-feedback controller to place the closed-loop poles at $p_1 = 0.9$, $p_2 = 0.9$ and $p_3 = 0.9$. Consider the case that in each sampling period, the tasks T_s, T_c and T_a are released respectively at $t = 0ms$, $t = 1.25ms$ and $t = 9.8ms$. Assume that the following assumption holds.

$$x[k+1] = \phi x[k] + \Gamma_1(D_c)u[k-1] + \Gamma_0(D_c)u[k]$$
$$\phi = e^{Ah} \approx I + Ah$$
$$\Gamma_1(D_c) \approx D_c B$$
$$\Gamma_0(D_c) = (h - D_c)B$$

Note: For simplicity, only steps for the computation of the feedback gain K should be shown. In particular,
(i) What is the value of γ_{aug}?
(ii) What is the value of $H(\phi_{aug})$?
(iii) What is K in terms of γ_{aug} and $H(\phi_{aug})$?

Solution 4(b)

The response times of task T_s, T_a and T_c are 0.4ms, 0.2ms and 3.4ms, so T_s can be finished before T_c starts and T_c finished before T_a. The sensor-to-actuator delay $D_c = 10$ms.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Correct A, B, C [10 marks, 0 if any one of them is incorrect]

$$\phi = e^{Ah} = \begin{bmatrix} 1 & h \\ h & h+1 \end{bmatrix} = \begin{bmatrix} 1 & 0.025 \\ 0.025 & 1.025 \end{bmatrix}, \Gamma = \begin{bmatrix} 0 \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 0.025 \end{bmatrix}$$

Correct numerical value of ϕ [10 marks]

$$\Gamma_1(D_c) = D_c B = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}$$

$$\Gamma_0(D_c) = (h - D_c)B = \begin{bmatrix} 0 \\ 0.015 \end{bmatrix}$$

Correct Γ_1 and Γ_0 [10 marks]

$$z[k] = \begin{bmatrix} x[k] \\ u[k-1] \end{bmatrix}$$

$$z[k+1] = \phi_{aug} z[k] + \Gamma_{aug} u[k], y[k] = C_{aug} z[k]$$

Correct $z[k]$ and $z[k+1]$ [5 marks]

$$\phi_{aug} = \begin{bmatrix} \phi & \Gamma_1(\tau) \\ 0 & 0 \end{bmatrix}, \Gamma_{aug} = \begin{bmatrix} \Gamma_0 \tau \\ I \end{bmatrix}, C_{aug} = \begin{bmatrix} C & 0 \end{bmatrix}$$

$$\phi_{aug} = \begin{bmatrix} \phi & \Gamma_1(D_c) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0.025 & 0 \\ 0.025 & 1.025 & 0.01 \\ 0 & 0 & 0 \end{bmatrix}, \Gamma_{aug} = \begin{bmatrix} \Gamma_0(D_c) \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0.015 \\ 1 \end{bmatrix}, C_{aug} = \begin{bmatrix} C & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Correct ϕ_{aug}, Γ_{aug} and C_{aug} [15 marks], 5 marks for each

$$\gamma_{aug} = \begin{bmatrix} \Gamma_{aug} & \phi_{aug}\Gamma_{aug} & \phi_{aug}^2\Gamma_{aug} \end{bmatrix} = \begin{bmatrix} 0 & 0.0004 & 0.001 \\ 0.015 & 0.0254 & 0.026 \\ 1 & 0 & 0 \end{bmatrix}$$

Correct numerical value of γ_{aug} [10 marks]

$$K = -\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \gamma_{aug}^{-1} H(\phi_{aug})$$

where

$$H(\phi_{aug}) = (\phi_{aug} - 0.9I)^3$$

Correct value of $H(\phi_{aug})$ [10 marks]

Correct value of K [5 marks]