Analytical Assignment - 3

Solutions

1. Let $q_i^{(j)}$ be the probability that output line i of stage j is idle, i.e., does not carry a memory request from a processor.
Define $p_i^{(j)} = 1 - q_i^{(j)}$

From the top output line of each stage-2 box, we can reach any of outputs 0, 1, 2, 3. The probability of a processor request to any of them is $1/2 + 3/14 = 10/14 = 5/7$. We therefore have,

$$q_i^{(2)} = \begin{cases} (2/7)^2 & \text{if } i = 0,1,2,3 \\ (5/7)^2 & \text{if } i = 4,5,6,7 \end{cases}$$

The corresponding value of $p_i^{(2)}$ is $45/49 = 0.918$ for $i = 0,1,2,3$ and $24/49 = 0.490$ for $i = 4,5,6,7$. Now consider input line i of a stage-1 box for $i = 0, 1, 2, 3$. If this line is carrying a request, the probability that such a request is to the top line of the box is : $\frac{1/2 + 1/14}{1/2 + 3/14} = 4/5$. The request is to either output line with equal probability for switch boxes with lines 4, 5, 6, 7 in stage-1. Hence,

$$q_i^{(1)} = \begin{cases} (1 - 4/5\, p_i^{(2)})^2 & \text{if } i = 0,1 \\ (1 - 1/5\, p_i^{(2)})^2 & \text{if } i = 2,3 \\ (1 - 1/2\, p_i^{(2)})^2 & \text{if } i = 4,5,6,7 \end{cases}$$

Substituting the values for $p_i^{(2)}$ yields :

$q_i^{(1)} = 0.071$; $p_i^{(1)} = 0.929$; for $i = 0, 1$

$q_i^{(1)} = 0.667$; $p_i^{(1)} = 0.333$; for $i = 2, 3$

$q_i^{(1)} = 0.570$; $p_i^{(1)} = 0.430$; for $i = 4, 5, 6, 7$

We reason similarly for stage-0 boxes. If an input line to the top box is busy, it will carry a request to output 0 with probability $\frac{1/2}{1/2 + 1/14} = 7/8$. Input lines to other boxes in this stage are equally likely to request either output line. Hence we can write:

$$q_i^{(0)} = \begin{cases} (1 - 7/8\ p_i^{(1)})^2 & \text{if } i = 0 \\ (1 - 1/8\ p_i^{(1)})^2 & \text{if } i = 1 \\ (1 - 1/2\ p_i^{(1)})^2 & \text{if } i = 2, 3, 4, 5, 6, 7 \end{cases}$$

The expected band width is $\sum_{i=0}^{7} p_i^{(0)} = 4 \times 0.384 + 2 \times 0.305 + 0.219 + 0.965$

$= 3.329$

2. Connectivity : $Q = 16 \cdot p_\ell^3$

Number of accessible processors : $Pr\{X^{(2)} = 1\}$
$= p_\ell [1 - (1 - p_\ell)^2]$

$A_r = 4 p_\ell Pr\{X^{(2)} = 1\}$.

Bandwidth : $BW = 4\ Pr\{X^{(0)} = 1\}$ where

$Pr\{X^{(2)} = 1\} = p_r$

$Pr\{X^{(1)} = 1\} = p_r p_\ell - \frac{1}{4} p_r^2 p_\ell^2$

$Pr\{X^{(0)} = 1\} = p_r p_\ell^2 - \frac{1}{4} p_r^2 p_\ell^3 - \frac{1}{4} p_\ell^2 (p_r p_\ell - \frac{1}{4} p_r^2 p_\ell^2)^2$.

In case 1, both A and B have to be operational; in case 2, one of them is not.
Hence, case 1 and 2 are disjoint.
Hence, case 1 and 2 are disjoint while this is not true for case 3.

In case 2, all links are good while this is not true for case 3.
This ensures that cases 2 and 3 are disjoint.
In case 1, both A and B are operational; in case 3, one of them is not and so cases 1 and 3 are disjoint.

For connectedness, consider nodes x, y.

Connectedness under case 1: Both A and B are connected and so if x and y are both in either A or in B, they have a path between them. If $x \in A$ and $y \in B$, then since at least one dimension-$(n-1)$ link between them is functional, we have a path that starts at x, transfers on a function dimension-$(n-1)$ link to B, then routes to y.

Connectedness under case 2: Suppose A is connected and B is not. Then, suppose z is the node in A connected by a dimension-$(n-1)$ edge to node y. Then the path from x to z and then to y exists.

Connectedness under case 3: If the dimension-$(n-1)$ edge zy (where z is defined as in case 2) is functional, the argument is identical to that used in case 2. If it is not, then there exists a link connecting y to some other node $w \in B$. Let $v \in A$ be the node connected to w along a dimension-$(n-1)$ link. We can therefore establish a path connecting y to w, connecting w to v and connecting v to x.

4. The six paths from node 0 to 7 are:

$P_1 = \{x_{0,1}, x_{1,3}, x_{3,7}\}$

$P_2 = \{x_{0,1}, x_{1,5}, x_{5,7}\}$

$P_3 = \{x_{0,2}, x_{2,3}, x_{3,7}\}$

$P_4 = \{x_{0,2}, x_{2,6}, x_{6,7}\}$

$P_5 = \{x_{0,4}, x_{4,5}, x_{5,7}\}$

$P_6 = \{x_{0,4}, x_{4,6}, x_{6,7}\}$

The corresponding probabilities are –

$T_1 = p_{0,1} p_{1,3} p_{3,7}$; $T_2 = p_{0,1} p_{1,5} p_{5,7} (1 - p_{1,3} p_{3,7})$;

$T_3 = p_{0,2} p_{2,3} p_{3,7} (q_{0,1} + p_{0,1} q_{1,3} (1 - p_{1,5} p_{5,7}))$;

$T_4 = p_{0,2} p_{2,6} p_{6,7} [q_{0,1} q_{2,3} + q_{0,1} p_{2,3} q_{3,7} + p_{0,1} q_{3,7} (1 - p_{1,5} p_{5,7}) + p_{0,1} p_{3,7}) 1 - p_{2,3}) (1 - p_{1,5} p_{5,7})]$;

T_5 and T_6 similarly.

Finally $R = T_1 + T_2 + T_3 + T_4 + T_5 + T_6$.