(1) $ADD = \{ x = y + z \mid x, y, z \in \{0,1\}^* \}$ over $\Sigma = \{0, 1, =\}$ is not regular.

$10^k = 1^k + 1$ — k pumping length

$|y| \geq 1$, $|xy| \leq k$ → consider "$10^k = 1^k + 1$" $= xyz$, all possible choice of x, y, z with $|y| \geq 1$ & $|xy| \leq k$

case a) $x = 10^a$, $y = 0^b$, $z = [0^{k-a-b} = 1^k + 1]$

$a \geq 0$, $b \geq 1$

pumping down: $10^a . (0^b)^0 \, 0^{k-a-b} = 1^{k+1}$

$\Rightarrow 10^{k-b} = 1^{k+1}$ resulting string $\notin ADD$

case b) x can be ϵ, $y = 10^a$, $z = [0^{k-a} = 1^{k+1}]$ continue

No other cases.

$\therefore$ for all possible x, y, z $\exists$ a way to pump a string $\notin ADD$

(2) $L = \{0^i 1^j \mid gcd(i, j) = 1\}$ is not regular.

pumping length = p, k is a prime no > p, choose $gcd(k, k-1!) = 1$

$0^k 1^{(k-1)!}$

$x = 0^a$, $y = 0^b$, $z = 0^{k-a-b} 1^{k-1!}$

$\therefore xy^0z = 0^a . 0^{k-a-b} . 1^{k-1!} = 0^{k-b} . 1^{k-1!}$

$b \geq 1$ — gcd = k−b

Properties,

Class of regular languages is not closed under homomorphism, inverse-homomorphism & substitution.

Sample application:

$L_1 \rightarrow \{0^n 1^n \mid n \geq 1\}$ is not regular → can use this to show

$L_2 = \{a^n b a^n \mid n \geq 1\}$ not regular.

if L_1 had a FA use it to simulate a for each 0.

Simulate ba for ~~1st~~ 1 seen.

Simulate a for each 1 thereafter.

This is not a proof of

$\therefore$ let us define the homomorphisms.

$h_1(a) = a$	$h_2(a) = 0$
$h_1(b) = ba$	$h_2(b) = 1$
$h_1(c) = a$	$h_2(c) = 1$

$$h_2\left(h_1^{-1}(\{a^n b a^n \mid n \geq 1\}) \cap a^* b c^*\right) = \{0^n 1^n \mid n \geq 1\}$$

$$(a+c)^n b (a+c)^{n-1} \cap a^* b c^*$$

$$h_2(a^n b c^{n-1})$$

$$0^n . 1 . 1^{n-1}$$

$$0^n 1^n$$

$\therefore$ if L_2 is regular, ~~that~~ makes $0^n 1^n$ also regular.

Quotients of Languages.

$$L_1 / L_2 = \{ x \mid \exists y \text{ in } L_2 \text{ so that } xy \in L_1 \}$$

Regular sets are closed under quotient with arbitrary sets.