

Multivariate Complexity Analysis of Geometric RED BLUE SET COVER

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Abstract We investigate the parameterized complexity of GENERALIZED RED BLUE SET COVER (GEN-RBSC), a generalization of the classic SET COVER problem and the more recently studied RED BLUE SET COVER problem. Given a universe Ucontaining b blue elements and r red elements, positive integers k_{ℓ} and k_r , and a family \mathcal{F} of ℓ sets over U, the GEN-RBSC problem is to decide whether there is a subfamily $\mathcal{F}' \subset \mathcal{F}$ of size at most k_{ℓ} that covers all blue elements, but at most k_r of the red elements. This generalizes SET COVER and thus in full generality it is intractable in the parameterized setting. In this paper, we study a geometric version of this problem, called GEN- RBSC- LINES, where the elements are points in the plane and sets are defined by lines. We study this problem for an array of parameters, namely, k_{ℓ}, k_{r}, r, b , and ℓ , and all possible combinations of them. For all these cases, we either prove that the problem is W-hard or show that the problem is fixed parameter tractable (FPT). In particular, on the algorithmic side, our study shows that a combination of k_{ℓ} and k_r gives rise to a nontrivial algorithm for GEN- RBSC- LINES. On the hardness side, we show that the problem is para-NP-hard when parameterized by k_r , and W[1]hard when parameterized by k_{ℓ} . Finally, for the combination of parameters for which GEN- RBSC- LINES admits FPT algorithms, we ask for the existence of polynomial

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kernels. We are able to provide a complete kernelization dichotomy by either showing that the problem admits a polynomial kernel or that it does not contain a polynomial kernel unless co-NP \subseteq NP/poly.

Keywords RBSC · FPT · W-hardness · Point line cover

1 Introduction

The input to a covering problem consists of a universe U of size n, a family \mathcal{F} of m subsets of U and a positive integer k, and the objective is to check whether there exists a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size at most k satisfying some desired properties. If \mathcal{F}' is required to contain all the elements of U, then it corresponds to the classical SET COVER problem. The SET COVER problem is part of Karp's 21 NP-complete problems [14]. This, together with its numerous variants, is one of the most well-studied problems in the area of algorithms and complexity. It is one of the central problems in all the paradigms that have been established to cope with NP-hardness, including approximation algorithms, randomized algorithms and parameterized complexity.

1.1 Problems Studied, Context and Framework

The goal of this paper is to study a generalization of a variant of SET COVER namely, the RED BLUE SET COVER problem.

RED BLUE SET COVER (RBSC) **Input:** A universe U = (R, B) where *R* is a set of *r* red elements and *B* is a set of *b* blue elements, a family \mathcal{F} of ℓ subsets of *U*, and a positive integer k_r . **Question:** Is there a subfamily \mathcal{F}' of sets that covers all blue elements but at most k_r red elements?

RED BLUE SET COVER was introduced in 2000 by Carr et al. [3]. The original motivation for studying the problem arose in the context of a data mining project [1] for detecting possible fraudulent data in Medicare claims. RED BLUE SET COVER can be thought of as a restricted version of the classification problem, which has applications in Machine Learning and Image Processing. It is also closely related to several combinatorial optimization problems such as the GROUP STEINER, MINIMUM LABEL PATH, MINIMUM MONOTONE SATISFYING ASSIGNMENT and SYMMETRIC LABEL COVER problems. See [3] for details. RED BLUE SET COVER is NP-complete, following from an easy reduction from SET COVER itself.

In this paper, we study the parameterized complexity, under various parameters, of a common generalization of both SET COVER and RED BLUE SET COVER, in a geometric setting.

GENERALIZED RED BLUE SET COVER (GEN-RBSC) **Input:** A universe U = (R, B) where *R* is a set of *r* red elements and *B* is a set of *b* blue elements, a family \mathcal{F} of ℓ subsets of *U*, and positive integers k_{ℓ}, k_r . **Question:** Is there a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size at most k_{ℓ} that covers all blue elements but at most k_r red elements?

It is easy to see that when $k_{\ell} = |\mathcal{F}|$ then the problem instance is a RED BLUE SET COVER instance, while it is a SET COVER instance when $k_{\ell} = k$, $R = \emptyset$, $k_r = 0$. Next we take a short detour and give a few essential definitions regarding parameterized complexity.

Parameterized Complexity The goal of parameterized complexity is to find ways of solving NP-hard problems more efficiently than brute force: here the aim is to restrict the combinatorial explosion to a parameter that is hopefully much smaller than the input size. Formally, a *parameterization* of a problem is assigning a positive integer parameter k to each input instance and we say that a parameterized problem is fixed-parameter tractable (FPT) if there is an algorithm that solves the problem in time $f(k) \cdot |I|^{\mathcal{O}(1)}$, where |I| is the size of the input and f is an arbitrary computable function depending only on the parameter k. Such an algorithm is called an FPT algorithm and such a running time is called FPT running time. There is also an accompanying theory of parameterized intractability using which one can identify parameterized problems that are unlikely to admit FPT algorithms. These are essentially proved by showing that the problem is hard for some class of the W-hierarchy. A parameterized problem is said to admit a h(k)-kernel if there is a polynomial time algorithm (the degree of the polynomial is independent of k), called a *kernelization* algorithm, that reduces the input instance to an equivalent instance whose size is upper bounded by h(k). If the function h(k) is polynomial in k, then we say that the problem admits a polynomial kernel. While positive kernelization results have appeared regularly over the last two decades, the first results establishing infeasibility of polynomial kernels for specific problems have appeared only recently. In particular, Bodlaender et al. [2], and Fortnow and Santhanam [12] have developed a framework for showing that a problem does not admit a polynomial kernel unless co-NP \subseteq NP/poly, which is deemed unlikely. For more background, the reader is referred to the following monograph [10].

In the parameterized setting, SET COVER, parameterized by k, is W[2]-hard [8] and it is not expected to have an FPT algorithm. The NP-hardness reduction from SET COVER to RED BLUE SET COVER implies that RED BLUE SET COVER is W[2]-hard parameterized by the size k_{ℓ} of a solution subfamily. However, the hardness result is not the end of the story for the SET COVER problem in parameterized complexity. In literature, various special cases of SET COVER have been studied. A few examples are instances with sets of bounded size [9], sets with bounded intersection [16,21], and instances where the bipartite incidence graph corresponding to the set family has bounded treewidth or excludes some graph H as a minor [5,11]. Apart from these results, there has also been extended study on different parameterizations of SET COVER. A special case of SET COVER which is central to the topic of this paper is the one where the *sets in the family correspond to some geometric object*. In the simplest geometric variant of SET COVER, called POINT LINE COVER, the elements of U are points in \mathbb{R}^2 and each set contains a maximal number of collinear points. This version of the problem is FPT and in fact has a polynomial kernel [16] of quadratic size. The size of these kernels have been proved to be tight, under standard assumptions, in [15]. When we take the sets to be the space bounded by unit squares, SET COVER is W[1]-hard [17]. On the other hand when surfaces of hyperspheres or hyperplanes are sets then the problem is FPT [16]. These geometric results motivate a systematic study of the parameterized complexity of geometric GEN-RBSC problems.

There is an array of natural parameters in hand for the GEN- RBSC problem. Hence, the problem promises an interesting dichotomy in parameterized complexity, under the various parameters. In this paper, we concentrate on the GENERALIZED RED BLUE SET COVER WITH LINES problem, parameterized under combinations of natural parameters.

GENERALIZED RED BLUE SET COVER WITH LINES (GEN-RBSC-LINES) **Input:** A universe U = (R, B) where *R* is a set of *r* red points and *B* is a set of *b* blue points, a family \mathcal{F} of ℓ subsets of *U* such that each subset contains a maximal set of collinear points of *U*, and positive integers k_{ℓ} , k_r .

Question: Is there a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size at most k_ℓ that covers all blue points but at most k_r red points?

To be more specific, a set $L \in \mathcal{F}$ is assiciated with a line equation, and the points in the set are all the points in U that satisfy the line equation. So, it is possible that a set in \mathcal{F} contains exactly one point from U. It is safe to assume that $r \ge k_r$, and $\ell \ge k_\ell$. Since it is enough to consider minimal solution families, we can also assume that $b \ge k_\ell$. This is because, by definition of minimality, each line in a minimal solution family covers a blue point that is not covered by any other line of the solution family.

We finish this section with some related results.

As mentioned earlier, the RED BLUE SET COVER problem in classical complexity is NP-complete. Interestingly, if the incidence matrix, built over the sets and elements, has the consecutive ones property then the problem is in P [6]. The problem has been studied in approximation algorithms as well [3,20]. Specially, the geometric variant, where every set is the space bounded by a unit square, has a polynomial time approximation scheme (PTAS) [4].

1.2 Our Contributions

In this paper, we first show a complete dichotomy of the parameterized complexity of GEN- RBSC- LINES. For a list of parameters, namely, k_{ℓ} , k_r , r, b, and ℓ , and all possible combinations of them, we show hardness or an FPT algorithm. Further, for parameterizations where an FPT algorithm exists, we either show that the problem admits a polynomial kernel or that it does not contain a polynomial kernel unless co-NP \subseteq NP/poly.

To describe our results we first state a few definitions. For a set $S \subseteq U$, we denote by 2^S the family of all the subsets of S, and by U^S the family of all the subsets of

Fig. 1 Illustration of our results for GEN- RBSC- LINES described in Theorem 1.1 and hierarchy of parameters



U that contain *S* (that is, all supersets of *S* in *U*). For a collection \mathfrak{F} of sets over a universe *U*, by DownClosure(\mathfrak{F}) and UpClosure(\mathfrak{F}) we mean the families $\bigcup_{S \in \mathfrak{F}} 2^S$ and $\bigcup_{S \in \mathfrak{F}} U^S$ respectively. Next, we describe the meaning of a parameterized result with respect to a set of parameters. If GEN- RBSC- LINES is FPT parameterized by a set Γ' of parameters this means that there exists an algorithm for GEN- RBSC- LINES parameterized by Γ' running in time $f(\Gamma') \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$. That is, the running time of the algorithm can depend in an arbitrary manner on the parameters present in Γ' . Equivalently, we have an algorithm running in time $f(\tau) \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$, where $\tau = \sum_{q \in \Gamma'} q$. Similarly, if the problem admits a polynomial kernel parameterized by Γ' then it means that there is no kernelization algorithm that outputs a kernel of size $\tau^{\mathcal{O}(1)}$ unless co-NP \subseteq NP/poly. Our first contribution is the following parameterized and kernelization dichotomy result for GEN- RBSC- LINES.

Theorem 1.1 Let $\Gamma = \{\ell, r, b, k_{\ell}, k_r\}$. Then, under the assumption that $FPT \subset W[1] \subset W[2] \subset \cdots \subset PARA-NP$, GEN-RBSC-LINES is FPT parameterized by $\Gamma' \subseteq \Gamma$ if and only if $\Gamma' \notin DownClosure(\{\{k_{\ell}, b\}, \{r, k_r\}\})$. Furthermore, unless co-NP \subseteq NP/poly, GEN-RBSC-LINES admits a polynomial kernel parameterized by $\Gamma' \subseteq \Gamma$ if and only if $\Gamma' \in UpClosure(\{\{\ell\}, \{k_{\ell}, r\}, \{b, r\}\})$.

A schematic diagram explaining the results proved in Theorem 1.1 can be seen in Fig. 1. Results for a $\Gamma' \subseteq \Gamma$ which is not depicted in Fig. 1 can be derived by checking whether Γ' is in DownClosure({ $\{k_{\ell}, b\}, \{r\}$ }). Let Γ' and Γ'' be two sets of parameters in Fig. 1. Let $\tau' = \sum_{q \in \Gamma'} q$ and $\tau'' = \sum_{q \in \Gamma''} q$. In Fig. 1, an arrow is given from Γ' to Γ'' if $\tau' \leq \tau''$ for all possible positive integral values that the parameters in Γ can take. In the parameterized setting, the arrows exhibit a lot of meaning regarding parameterized results. If GEN- RBSC- LINES parameterized by Γ' has an FPT algorithm then the same algorithm is an FPT algorithm for GEN- RBSC-LINES parameterized by Γ'' . On the other hand, if GEN- RBSC- LINES parameterized by Γ'' does not have a polynomial kernel, under standard assumptions, then GEN-RBSC- LINES parameterized by Γ' also does not have a polynomial kernel under those assumptions. Similarly, if GEN- RBSC- LINES parameterized by Γ'' does not belong to the FPT class, under standard assumptions, then GEN- RBSC- LINES parameterized by Γ'' also does not belong to the FPT class under those assumptions.

Next we consider the RBSC- LINES problem. Here we do not have any constraint on how many sets we pick in the solution family but we are still allowed to cover at Fig. 2 Illustration of our results for RED BLUE SET COVER WITH LINES under various parameters



most k_r red points. This brings two main changes in Fig. 1. For GEN- RBSC- LINES we show that the problem is NP-hard even when there is a constant number of red points. However, RBSC- LINES becomes FPT parameterized by r. In contrast, RBSC- LINES is W[2]-hard parameterized by k_r . This leads to the following dichotomy theorem for RBSC- LINES.

Theorem 1.2 Let $\Gamma = \{\ell, r, b, k_r\}$. Then, under the assumption that $FPT \subset W[1] \subset W[2] \subset \ldots \subset PARA- NP$, RBSC- LINES is FPT parameterized by $\Gamma' \subseteq \Gamma$ if and only if $\Gamma' \notin \{\{b\}, \{k_r\}\}$. Furthermore, unless co-NP \subseteq NP/poly, RBSC- LINES admits polynomial kernel parameterized by $\Gamma' \subseteq \Gamma$ if and only if $\Gamma' \in UpClosure(\{\{\ell\}, \{b, r\}\})$.

A schematic diagram explaining the results proved in Theorem 1.2 is given in Fig. 2.

A quick look at Fig. 1 will show that the GEN- RBSC- LINES problem is FPT parameterized by $k_{\ell} + k_r$ or $b + k_r$. A natural question to ask is whether GEN- RBSC itself (the problem where sets in the input family are arbitrary and do not correspond to lines) is FPT when parameterized by $b + k_r$. Regarding this, we show the following results:

- 1. GEN- RBSC is W[1]-hard parameterized by $k_{\ell} + k_r$ (or $b + k_r$) when every set has size at most three and contains at least two red points.
- 2. GEN- RBSC is W[2]-hard parameterized by $k_{\ell} + r$ when every set contains at most one red point.

The first result essentially shows that GEN- RBSC is W[1]-hard even when the sets in the family have size *bounded by three*. This is in sharp contrast to SET COVER, which is known to be FPT parameterized by k_{ℓ} and d. Here, d is the size of the maximum cardinality set in \mathcal{F} . In fact, SET COVER admits a kernel of size $k_{\ell}^{\mathcal{O}(d)}$. This leads to the following question:

Does the hardness of GEN- RBSC in item one arise from the presence of two red points in the instance? Would the complexity change if we assume that each set contains at most one red point?

In fact, even if we assume that each set contains at most one red point, we must take *d*, the size of the maximum cardinality set in \mathcal{F} , as a parameter. Else, this would correspond to the hardness result presented in item two. As a final algorithmic result we show that GEN- RBSC admits an algorithm with running time $2^{\mathcal{O}(dk_\ell)} \cdot (|U|+|\mathcal{F}|)^{\mathcal{O}(1)}$, when every set has at most one red point. Observe that in this setting k_r can always be assumed to be less than k_ℓ . Thus, this is also a FPT algorithm parameterized by $k_\ell + k_r$,

when sets in the input family have bounded size. However, we show that GEN- RBSC (in fact GEN- RBSC- LINES) does not admit a polynomial kernel parameterized by $k_{\ell} + k_r$ even when each set in the input family corresponds to a line and has size two and contains at most one red point.

1.3 Our Methods and an Overview of Main Algorithmic Results

Let $\Gamma = \{\ell, r, b, k_\ell, k_r\}$. Most of our W-hardness results for a GEN-RBSC variant parameterized by $\Gamma' \subseteq \Gamma$ are obtained by giving a polynomial time reduction, from SET COVER or MULTICOLORED CLIQUE that makes every $q \in \Gamma'$ at most $k^{\mathcal{O}(1)}$ (in fact most of the time $\mathcal{O}(k)$).

This allows us to transfer the known hardness results about SET COVER and MUL-TICOLORED CLIQUE to our problem. Since in most cases the parameters are linear in the input parameter, in fact we can rule out an algorithm of the form $(|U| + |\mathcal{F}|)^{o(\tau)}$, where $\tau = \sum_{q \in \Gamma'} q$, under the Exponential Time Hypothesis (ETH) [13]. Similarly, hardness results for kernels are derived from giving an appropriate polynomial time reduction from parameterized variants of the SET COVER problem that only allows each parameter $q \in \Gamma'$ to grow polynomially in the input parameter.

Our main algorithmic highlights are parameterized algorithms for

- (a) GEN- RBSC- LINES running in time $2^{\mathcal{O}(k_{\ell} \log k_{\ell} + k_r \log k_r)} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$ (showing GEN- RBSC- LINES is FPT parameterized by $k_{\ell} + k_r$); and
- (b) GEN- RBSC with running time $2^{\mathcal{O}(dk_{\ell})} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$, when every set is of size at most *d* and has at most one red point.

Observe that the first algorithm generalizes the known algorithm for POINT LINE COVER which runs in time $2^{\mathcal{O}(k_{\ell} \log k_{\ell})} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$ [16].

The parameterized algorithm for GEN-RBSC-LINES mentioned in (a) starts by bounding the number of blue points by k_{ℓ}^2 and guessing the lines that contain at least two blue points. The number of lines containing at least two blue points can be shown to be at most k_{ℓ}^4 . These guesses lead to an equivalent instance where each line contains exactly one blue point and there are no lines that only contain red points (as these lines can be deleted). However, we can not bound the number of red points at this stage. We introduce a notion of "solution subfamily" and connected components of the solution subfamilies. Interestingly, this equivalent instance has sufficient geometric structure on the connected components. We exploit the structure of these components, gotten mainly from simple properties of lines on a plane, to show that knowing one of the lines in each component can, in FPT time, lead to finding the component itself! Thus, to find a component all we need to do is to guess one of the lines in it. However, here we face our second difficulty: the number of connected components can be as bad as $\mathcal{O}(k_{\ell})$ and thus if we guess one line for each connected component then it would lead to a factor of $|\mathcal{F}|^{\mathcal{O}(k_{\ell})}$ in the running time of the algorithm. However, our equivalent instances are such that we are allowed to process each component independent of other components. This brings the total running time of guessing the first line of each component down to $k_{\ell} \cdot |\mathcal{F}|$. The algorithmic ideas used here can be viewed as some sort of "geometry preserving subgraph isomorphism", which could be useful in other

contexts also. This completes an overview of the FPT result for GEN- RBSC- LINES parameterized by $k_{\ell} + k_r$.

The algorithm for GEN- RBSC running in time $2^{\mathcal{O}(dk_{\ell})} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$, where every set is of size at most *d* and has at most one red point is purely based on a novel reduction to SUBGRAPH ISOMORPHISM where the subgraph we are looking for has size $\mathcal{O}(k_{\ell}d)$ and treewidth 3. The host graph, where we are looking for a solution subgraph, is obtained by starting with the bipartite incidence graph and making modifications to it. The bipartite incidence graph we start with has in one side vertices for sets and in the other side vertices corresponding to blue and red points and there is an edge between vertices corresponding to a set and a blue (red) point if this blue (red) point is contained in the set. Our main observation is that a solution subfamily can be captured by a subgraph of size $\mathcal{O}(k_{\ell}d)$ and treewidth 3. Thus, for our algorithm we enumerate all such subgraphs in time $2^{\mathcal{O}(dk_{\ell})} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$ and for each such subgraph we check whether it exists in the host graph using known algorithms for SUBGRAPH ISOMORPHISM. This concludes the description of this algorithm.

2 Preliminaries

In this paper an undirected graph is denoted by a tuple G = (V, E), where V denotes the set of vertices and E the set of edges. For a set $S \subseteq V$, the subgraph of G induced by S, denoted by G[S], is defined as the subgraph of G with vertex set S and edge set $\{(u, v) \in E : u, v \in S\}$. The subgraph obtained after deleting S is denoted as $G \setminus S$. All vertices adjacent to a vertex v are called neighbors of v and the set of all such vertices is called the neighborhood of v. Similarly, a non-adjacent vertex of v is called a non-neighbor and the set of all non-neighbors of v is called the non-neighborhood of v. The neighborhood of v is denoted by N(v). A vertex in a connected graph is called a cut vertex if its deletion results in the graph becoming disconnected.

Recall that showing a problem W[1] or W[2] hard implies that the problem is unlikely to be FPT. One can show that a problem is W[1]-hard (W[2]-hard) by presenting a parameterized reduction from a known W[1]-hard problem (W[2]-hard) such as CLIQUE (SET COVER) to it. The most important property of a parameterized reduction is that it corresponds to an FPT algorithm that bounds the parameter value of the constructed instance by a function of the parameter of the source instance. A parameterized problem is said to be in the class para-NP if it has a nondeterministic algorithm with FPT running time. To show that a problem is para-NP-hard we need to show that the problem is NP-hard for some constant value of the parameter. For example, *k*- COLORING is para-NP-hard parameterized by the number of colors, since the problem is NP-complete when k = 3. See [10] for more details.

Lower Bounds in Kernelization In recent years, several techniques have been developed to show that certain parameterized problems belonging to the FPT class cannot have any polynomial sized kernel unless some classical complexity assumptions are violated. One such technique that is widely used is the polynomial parameter transformation technique. **Definition 2.1** [2] Let Π , Γ be two parameterized problems. A polynomial time algorithm \mathcal{A} is called a polynomial parameter transformation (or ppt) from Π to Γ if, given an instance (x, k) of Π , \mathcal{A} outputs in polynomial time an instance (x', k') of Γ such that $(x, k) \in \Pi$ if and only if $(x', k') \in \Gamma$ and $k' \leq p(k)$ for a polynomial p.

We use the following theorem together with ppt reductions to rule out polynomial kernels.

Theorem 2.1 [2] Let Π , Γ be two parameterized problems such that Π is NP-hard and $\Gamma \in NP$. Assume that there exists a polynomial parameter transformation from Π to Γ . Then, if Π does not admit a polynomial kernel neither does Γ .

For further details on lower bound techniques in kernelization refer to [2, 12].

Generalized Red Blue Set Cover A set *S* in a GENERALIZED RED BLUE SET COVER instance (U, \mathcal{F}) is said to *cover* a point $p \in U$ if $p \in S$. A *solution family* for the instance is a family of sets of size at most k_{ℓ} that covers all the blue points and at most k_r red points. In case of RED BLUE SET COVER, the solution family is simply a family of sets that covers all the blue points but at most k_r red points. Such a family will also be referred to as a *valid family*. A *minimal family of sets* is a family of sets such that every set contains a unique blue point. In other words, deleting any set from the family implies that a strictly smaller set of blue points is covered by the remaining sets. The sets of GENERALIZED RED BLUE SET COVER WITH LINES are also called *lines* in this paper. We also mention a key observation about lines in this section. This observation is crucial in many arguments in this paper.

Observation 1 Given a set of points S, let \mathcal{F} be a set of lines such that each line contains at least 2 points from S. Then $|\mathcal{F}| \leq {|S| \choose 2}$.

This observation comes from the fact that any two lines can intersect at exactly 1 point. Therefore, any subset consisting of 2 points from *S* is contained in at most one line of \mathcal{F} .

GEN- RBSC with hyperplanes of \mathbb{R}^d , for a fixed positive integer d, is a special case of the problem. Here, the input universe U is a set of n points in \mathbb{R}^d . A hyperplane in \mathbb{R}^d is the affine hull of a set of d affinely independent points [16]. In our special case each set is a maximal set of points that lie on a hyperplane of \mathbb{R}^d .

Definition 2.2 An intersection graph $G_{\mathcal{F}} = (V, E)$ for an instance (U, \mathcal{F}) of GEN-ERALIZED RED BLUE SET COVER is a graph with vertices corresponding to the sets in \mathcal{F} . We give an edge between two vertices if the corresponding sets have non-empty intersection.

The following proposition is a collection of results on the SET COVER problem, that will be repeatedly used in the paper. The results are from [7,8]

Proposition 1 The SET COVER problem is:

- (i) W[2] hard when parameterized by the solution family size k.
- (ii) FPT when parameterized by the universe size n, but does not admit polynomial kernels unless co-NP \subseteq NP/poly.

(iii) FPT when parameterized by the number of sets m in the instance, but does not admit polynomial kernels unless $co-NP \subseteq NP/poly$.

Tree Decompositions and Treewidth We also need the concept of treewidth and tree decompositions.

Definition 2.3 (*Tree Decomposition* [22]) A tree decomposition of a (undirected or directed) graph G = (V, E) is a tree \mathbb{T} in which each vertex $x \in \mathbb{T}$ has an assigned set of vertices $B_x \subseteq V$ (called a bag) such that $(\mathbb{T}, \{B_x\}_{x \in \mathbb{T}})$ has the following properties:

- $-\bigcup_{x\in\mathbb{T}}B_x=V$
- For any $(u, v) \in E$, there exists an $x \in \mathbb{T}$ such that $u, v \in B_x$.
- If $v \in B_x$ and $v \in B_y$, then $v \in B_z$ for all z on the path from x to y in \mathbb{T} .

In short, we denote $(\mathbb{T}, \{B_x\}_{x \in \mathbb{T}})$ as \mathbb{T} .

The *treewidth* $tw(\mathbb{T})$ of a tree decomposition \mathbb{T} is the size of the largest bag of \mathbb{T} minus one. A graph may have several distinct tree decompositions. The treewidth tw(G) of a graph G is defined as the minimum of treewidths over all possible tree decompositions of G.

3 Parameterizing by k_r and r

In this section we first show that GEN- RBSC- LINES parameterized by r is para-NP-complete. Since $k_r \le r$, it follows that GEN- RBSC- LINES parameterized by k_r is also para-NP-complete.

Theorem 3.1 GEN- RBSC- LINES is para-NP-complete parameterized by either r or k_r .

Proof If we are given a solution family for an instance of GEN- RBSC- LINES we can check in polynomial time if it is valid.

Hence, GEN- RBSC- LINES has a nondeterministic algorithm with FPT running time (in fact polynomial) and thus GEN- RBSC- LINES parameterized by r is in para-NP.

For completeness, there is an easy polynomial-time many-one reduction from the POINT LINE COVER problem, which is NP-complete. An instance $((U, \mathcal{F}))$ of POINT LINE COVER parameterized by k, the size of the solution family, is reduced to an instance $((R \cup B, \mathcal{F}))$ of GEN-RBSC-LINES parameterized by r or k_r with the following properties:

- -B = U
- The family of sets remains the same in both instances.
- R consists of 1 red vertex that does not belong to any of the lines of \mathcal{F} .
- $-k_{\ell} = k$ and $k_r = 0$.

It is easy to see that $((U, \mathcal{F}))$ is a YES instance of POINT LINE COVER if and only if $(R \cup B, \mathcal{F})$ is a YES instance of GEN- RBSC- LINES. Since the reduced instances belong to GEN- RBSC- LINES parameterized by r = 1 or $k_r = 0$, this proves that GEN- RBSC- LINES parameterized by r or k_r is para-NP-complete.

4 Parameterizing by *l*

In this section we design a parameterized algorithm as well as a kernel for GEN-RBSC-LINES when parameterized by the size ℓ of the family. The algorithm for this is simple. We enumerate all possible k_{ℓ} -sized subsets of input lines and for each subset, we check in polynomial time whether it covers all blue points and at most k_r red points. The algorithm runs in time $\mathcal{O}(2^{\ell} \cdot (|U| + |\mathcal{F}|))$. The main result of this section is a polynomial kernel for GEN-RBSC- LINES when parameterized by ℓ .

We start by a few reduction rules which will be used not only in the kernelization algorithm given below but also in other parameterized and kernelization algorithms in subsequent sections.

Reduction Rule 1 If there is a set $S \in \mathcal{F}$ with only red points then delete S from \mathcal{F} .

Lemma 4.1 Reduction Rule 1 is safe.

Proof Let \mathcal{F}' be a family of at most k_{ℓ} lines of the given instance that cover all blue points and at most k_r red points. If \mathcal{F}' contains S, then $\mathcal{F}' \setminus \{S\}$ is also a family of at most k_{ℓ} lines that cover all blue points and at most k_r red points. Hence, we can safely delete S. This shows that Reduction Rule 1 is safe.

Reduction Rule 2 If there is a set $S \in \mathcal{F}$ with more than k_r red points in it then delete *S* from \mathcal{F} .

Lemma 4.2 Reduction Rule 2 is safe.

Proof If S has more than k_r red points then S alone exceeds the budget given for the permissible number of covered red points. Hence, S cannot be part of any solution family and can be safely deleted from the instance. This shows that Reduction Rule 2 is safe.

Our final rule is as follows. A similar Reduction Rule was used in [16], for the POINT LINE COVER problem.

Reduction Rule 3 If there is a set $S \in \mathcal{F}$ with at least $k_{\ell} + 1$ blue points then reduce the budget of k_{ℓ} by 1 and the budget of k_r by $|R \cap S|$. The new instance is $(U \setminus S, \tilde{\mathcal{F}})$, where $\tilde{\mathcal{F}} = \{F \setminus S \mid F \in \mathcal{F} \text{ and } F \neq S\}$.

Lemma 4.3 Reduction Rule 3 is safe.

Proof If *S* is not part of the solution family then we need at least $k_{\ell} + 1$ lines in the solution family to cover the blue points in *S*, which is not possible. Hence any solution family must contain *S*.

Suppose the reduced instance has a solution family \mathcal{F}' covering $B \setminus S$ blue points and at most $k_r - |R \cap S|$ red points from $R \setminus S$. Then $\mathcal{F}' \cup \{S\}$ is a solution for the original instance. On the other hand, suppose the original instance has a solution family $\hat{\mathcal{F}}$. As argued above, $S \in \hat{\mathcal{F}}$. $\hat{\mathcal{F}} \setminus S$ covers all blue points of $B \setminus S$ and at most $k_r - |R \cap S|$ red points from $R \setminus S$, and is a candidate solution family for the reduced instance. Thus, Reduction Rule 3 is safe. The following simple observation can be made after exhaustive application of Reduction Rule 3. This is similar to the construction of the quadratic kernel for POINT LINE COVER [16].

Observation 2 If the budget for the subfamily \mathcal{F}' to cover all blue and at most k_r red points is k_ℓ then after exhaustive applications of Reduction Rule 3 there can be at most k_ℓ^2 blue points remaining in a YES instance. If there are more than k_ℓ^2 blue points remaining to be covered then we correctly say NO.

It is worth mentioning that even if we had integer weights on the red points in R and asked for a solution family of size at most k_{ℓ} that covered all blue points but red points of total weight at most k_r , then this weighted version, called WEIGHTED GEN- RBSC- LINES parameterized by ℓ is FPT. The WEIGHTED GEN- RBSC- LINES problem will be useful in the theorem below. Finally, we get the following result.

Theorem K.1 There exists a polynomial kernel for GEN- RBSC- LINES parameterized by ℓ . Also, there is an algorithm for GEN- RBSC- LINES running in time $\mathcal{O}(2^{\ell} \cdot (|U| + |\mathcal{F}|))$.

Proof We have already described the enumeration based algorithm at the beginning of this section. Here, we only give a non-constructive proof that a polynomial kernel exists. Given an instance of GEN- RBSC- LINES we exhaustively apply Reduction Rules 1, 2 and 3 to obtain an equivalent instance. By Observation 2 and the fact that $k_{\ell} \leq \ell$, the current instance must have at most ℓ^2 blue points, or we can safely say NO. Also, the number of red points that belong to 2 or more lines is bounded by the number of intersection points of the ℓ lines, i.e., ℓ^2 . Any remaining red points belong to exactly 1 line. We reduce our GEN- RBSC- LINES instance to a WEIGHTED GEN- RBSC- LINES instance as follows:

- The family of lines and the set of blue points remain the same in the reduced instance. The red points appearing in the intersection of two lines also remain the same. Give a weight of 1 to these red points.
- For each line L, let c(L) indicate the number of red points that belong exclusively to L. Remove all but one of these red points and give weight c(L) to the remaining exclusive red point.

In the WEIGHTED GEN- RBSC- LINES instance, there are ℓ lines, at most ℓ^2 blue points and at most $\ell^2 + \ell$ red points. For each line *L*, the value of c(L) is at most k_r , after Reduction Rule 2. Suppose $k_r > 2^{\ell}$. Then $r > 2^{\ell}$ and the parameterized algorithm for GEN- RBSC- LINES running in time $\mathcal{O}(2^{\ell} \cdot (|U| + |\mathcal{F}|))$ runs in polynomial time. Thus we can assume that $k_r \leq 2^{\ell}$. Then we can represent k_r and therefore the weights c(L) by at most ℓ bits. Thus, the reduced instance has size bounded by $\mathcal{O}(\ell^2)$.

Observe that we got an instance of WEIGHTED GEN-RBSC-LINES and not of GEN-RBSC-LINES which is the requirement for the kernelization procedure. All this shows is that the reduction is a "compression" from GEN-RBSC-LINES parameterized by ℓ to WEIGHTED GEN-RBSC-LINES parameterized by ℓ . This is rectified by the following standard procedure. Since both problems belong to NP, there is a polynomial time many-one reduction from WEIGHTED GEN-RBSC-LINES to GEN-RBSC-LINES.

Finally, using this polynomial time reduction, we obtain a polynomial size kernel for GEN- RBSC- LINES parameterized by ℓ . This completes a non-constructive proof of the existence of a polynomial kernel for GEN- RBSC- LINES.

Observe that the algorithm referred to in Theorem K.1 does not use the fact that sets are lines and thus it also works for GEN- RBSC parameterized by ℓ . However, it follows from Proposition 1(iii) that GEN- RBSC parameterized by ℓ does not admit a polynomial kernel.

5 Parameterizing by k_{ℓ} , *b* and $k_{\ell} + b$

In this section we look at GEN- RBSC- LINES parameterized by k_{ℓ} , b, and $k_{\ell} + b$. There is an interesting connection between b and k_{ℓ} . As we are looking for minimal solution families, we can alway assume that $b \ge k_{\ell}$. On the other hand, Reduction Rule 3 showed us that for all practical purposes $b \le k_{\ell}^2$. Thus, in the realm of parameterized complexity k_{ℓ} , b and $k_{\ell} + b$ are the *same parameters*. That is, GEN- RBSC- LINES is FPT parameterized by k_{ℓ} if and only if it is FPT parameterized by b if and only if it is FPT parameterized by $k_{\ell} + b$. The same holds in the context of kernelization complexity.

First, we show that GEN- RBSC- LINES parameterised by k_{ℓ} or *b* is W[1]-hard. Then we look at some special cases that turn out to be FPT.

5.1 Parameter $k_{\ell} + b$

We look at GEN- RBSC- LINES parameterized by $k_{\ell} + b$. This problem is not expected to have a FPT algorithm as it is W[1]-hard. We give a reduction to this problem from the MULTICOLORED CLIQUE problem, which is known to be W[1] hard even on regular graphs [19].

MULTICOLORED CLIQUE **Parameter:** *k* **Input:** A graph G = (V, E) where $V = V_1 \uplus V_2 \uplus \ldots \uplus V_k$, and an integer *k*. **Question:** Is there a clique $C \subseteq G$ of size *k* such that $\forall 1 \le i \le k, C \cap V_i \ne \emptyset$.

The partition of the vertex set can be considered as a coloring on the vertex set and a clique containing one vertex from each part is called a *multi-colored clique*.

Theorem 5.1 GEN- RBSC- LINES parameterized by k_{ℓ} or b or $k_{\ell} + b$ is W[1]-hard.

Proof We will give a reduction from MULTICOLORED CLIQUE on regular graphs. Let (G = (V, E), k) be an instance of MULTICOLORED CLIQUE, where *G* is a *d*-regular graph with *n* vertices and *m* edges. We construct an instance of GEN-RBSC-LINES $(R \cup B, \mathcal{F})$, as follows. Let $V = V_1 \uplus V_2 \uplus \cdots \uplus V_k$. A multi-colored clique cannot contain an edge whose both end points are in the same vertex partition. Therefore, without loss of generality, assume that each V_i , $1 \le i \le k$ is an independent set.

- 1. For each vertex class V_i , $1 \le i \le k$, add two blue points b_i at (0, i) and b'_i at (i, 0).
- 2. Informally, for each vertex class V_i , $1 \le i \le k$ we do as follows. Let L_k be the line that is parallel to y axis and passes through the point (k, 0). Suppose there are n_i vertices in V_i . We select n_i distinct points, say \mathcal{P} , in \mathbb{R}^2 on the line L_k , such that for all $p \in \mathcal{P}$ with coordinates (a_1, a_2) , $a_1 = k$ (as these are points on L_k) and a_2 lies in the interval $(i 1, i \frac{1}{2})$. Now for every point $p \in \mathcal{P}$ we draw the unique line between (0, i) and the point p. Finally, we assign each line to a unique vertex in V_i . Formally, we do as follows. For each vertex class V_i , $1 \le i \le k$ and each vertex $u \in V_i$, we choose a point $p_u^1 \in \mathbb{R}^2$ with coordinates (k, y_u) , $i 1 < y_u < i \frac{1}{2}$. Also, for each pair $u \neq v \in V_i$, $y_u \neq y_v$. For each $u \in V_i$, we add the line l_u^1 , defined by b_i and p_u^1 , to \mathcal{F} . We call these *near-horizontal lines*. Observe that all the near-horizontal lines corresponding to vertices in V_i intersect at b_i . Furthermore, for any two vertices $u \in V_i$ and $v \in V_j$, with $i \neq j$, the lines l_u^1 and l_v^1 do not intersect on a point with *x*-coordinate from the closed interval [0, k].
- 3. Similarly, for each vertex class V_i , $1 \le i \le k$ and each vertex $u \in V_i$, we choose a point $p_u^2 \in \mathbb{R}^2$ with coordinates (x_u, k) , $i 1 < x_u < i \frac{1}{2}$. Again, for each pair $u \ne v \in V_i$, $x_u \ne x_v$. For each $u \in V_i$, we add the line l_u^2 , defined by b'_i and p_u^2 , to \mathcal{F} . Notice that for any $u, v \in V$, l_u^1 and l_v^2 have a non-empty intersection. We call these *near-vertical lines*. Observe that all the near-vertical lines corresponding to vertices in V_i intersect at b'_i . Furthermore, for any two vertices $u \in V_i$ and $v \in V_j$, with $i \ne j$, the lines l_u^2 and l_v^2 do not intersect on a point with y-coordinate from the closed interval [0, k]. However, a near-horizontal line and a near-vertical line will intersect at a point with both x and y-coordinate from the closed interval [0, k]. The construction ensures that no 3 lines in \mathcal{F} have a common intersection.
- 4. For each edge $e = (u, v) \in E$, add two red points, r_{uv} at the intersection of lines l_u^1 and l_v^2 , and r_{vu} at the intersection of lines l_v^1 and l_u^2 .
- 5. For each vertex $v \in V$, add a red point at the intersection of the lines l_v^1 and l_v^2 .

This concludes the description of the reduced instance. Thus we have an instance $(R \cup B, \mathcal{F})$ of GEN- RBSC- LINES with 2n lines, 2k blue points and 2m + n red points.

Claim G = (V, E) has a multi-colored clique of size k if and only if $(R \cup B, \mathcal{F})$ has a solution family of 2k lines, covering the 2k blue points and at most $2(d + 1)k - k^2$ red points.

Proof Assume there exists a multi-colored clique *C* of size *k* in *G*. Select the 2*k* lines corresponding to the vertices in the clique. That is, select the subset of lines $\mathcal{F}' = \{l_u^j \mid 1 \le j \le 2, u \in C\}$ in the GEN- RBSC- LINES instance. Since the clique is multi-colored, these lines cover all the blue points. Each line (near-horizontal or near-vertical) covers exactly d + 1 red points. Thus, the number of red points covered by \mathcal{F}' is at most (d + 1)2k. However, each red point corresponding to vertices in *C* and the two red points covered by \mathcal{F}' is at most (d + 1)2k. However, each edge in *C* are counted twice. Thus, the number of red points covered by \mathcal{F}' is at most $(d + 1)2k - k - 2\binom{k}{2} = 2(d + 1)k - k^2$. This completes the proof in the forward direction.

Now, assume there is a minimal solution family of size at most 2k, containing at most $2(d + 1)k - k^2$ red points. As no two blue points are on the same line and

there are 2k blue points, there exists a unique line covering each blue point. Let \mathcal{L}^1 and \mathcal{L}^2 represent the sets of near-horizontal and near-vertical lines respectively in the solution family. Observe that \mathcal{L}^1 covers $\{b_1, \ldots, b_k\}$ and \mathcal{L}^2 covers $\{b'_1, \ldots, b'_k\}$. Let $C = \{v_1, \ldots, v_k\}$ be the set of vertices in *G* corresponding to the lines in \mathcal{L}^1 . We claim that *C* forms a multicolored *k*-clique in *G*. Since b_i can only be covered by lines corresponding to the vertices in V_i and \mathcal{L}^1 covers $\{b_1, \ldots, b_k\}$ we have that $C \cap V_i \neq \emptyset$. It remains to show that for every pair of vertices in *C* there exists an edge between them in *G*. Let v_i denote the vertex in $C \cap V_i$.

Consider all the lines in \mathcal{L}^1 . Each of these lines are near-horizontal and covers exactly d + 1 red points. Furthermore, no two of them intersect at a red point. Since \mathcal{L}^1 and \mathcal{L}^2 together can cover at most $2k(d + 1) - k^2$ red points in total, at least k^2 of them must be covered by both sets. Since each set has exactly k lines (and the lines are distinct), any two lines l_1 and l_2 such that $l_1 \in \mathcal{L}^1$ and $l_2 \in \mathcal{L}^2$ must intersect at a red point. This implies that either l_1 and l_2 correspond to the same vertex in V or there exists an edge between the vertices corresponding to them. Let $C' = \{w_1, \ldots, w_k\}$ be the set of vertices in G corresponding to the lines in \mathcal{L}^2 . Since b'_i can only be covered by lines corresponding to the vertices in V_i and \mathcal{L}^2 covers $\{b'_1, \ldots, b'_k\}$ we have that $C' \cap V_i \neq \emptyset$. Let w_i denote the vertex in V_i such that $l^2_{w_i} \in \mathcal{L}^2$ covers b'_i . We know that $l^1_{w_i}$ and $l^2_{w_i}$ must intersect on a red point. However, by construction no two distinct vertices v_i and w_i belonging to the same vertex class V_i intersect at red point. Thus $v_i = w_i$. This means C = C'. This, together with the fact that two lines l_1 and l_2 such that $l_1 \in \mathcal{L}^1$ and $l_2 \in \mathcal{L}^2$ (now lines corresponding to C) must intersect at a red point. multicolored k-clique in G.

Since $b = k_{\ell} = 2k$, we have that GEN- RBSC- LINES is W[1]-hard parameterized by k_{ℓ} or b or $k_{\ell} + b$. This concludes the proof.

A closer look at the reduction shows that every set contains exactly one blue point. A natural question to ask is whether the complexity would change if we take the complement of this scenario, that is, each set contains either no blue points or at least two blue points. Shortly, we will see that this implies that the problem becomes FPT. Also, notice that each set in the reduction contains unbounded number of red elements. What about the parameterized complexity if every set in the input contained at most a bounded number, say *d*, of red elements. Even then the complexity would change but for this we need an algorithm for GEN- RBSC- LINES parameterized by $k_{\ell} + k_r$ that will be presented in Sect. 6.

5.2 Special Case Under the Parameter k_{ℓ}

In this section, we look at the special case when every line in the GEN- RBSC- LINES instance contains at least 2 blue points or no blue points at all. We show that in this restricted case GEN- RBSC- LINES is FPT.

Theorem K.2 GEN- RBSC- LINES parameterized by k_{ℓ} , where input instances have each set containing either at least 2 blue points or no blue points, has a polynomial kernel. There is also an FPT algorithm running in $\mathcal{O}(k_{\ell}^{4k_{\ell}} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)})$ time. *Proof* We exhaustively apply Reduction Rules 1, 2 and 3 to our input instance. In the end, we obtain an equivalent instance that has at least 1 blue point per line. The equivalent instance also has each line containing at least 2 blue points or no blue points. The instance has at most $b = k_{\ell}^2$ blue points, or else we can correctly say NO. By Observation 1 and the assumption on the instance, we can bound ℓ by $\binom{b}{2} \le k_{\ell}^4$. Now from Theorem K.1 we get a polynomial kernel for this special case of GEN- RBSC-LINES parameterized by k_{ℓ} .

Regarding the FPT algorithm, we are allowed to choose at most k_{ℓ} solution lines from a total of $\ell \leq k_{\ell}^4$ lines in the instance (of course after we have applied Reduction Rules 1, 2 and 3 exhaustively). For every possible k_{ℓ} -sized set of lines we check whether the set covers all blue vertices and at most k_r red vertices. If the instance is a YES instance, one such k_{ℓ} -sized set is a solution family. This algorithm runs in $\mathcal{O}(\binom{k_{\ell}^4}{k_{\ell}}) \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}) = \mathcal{O}(k_{\ell}^{4k_{\ell}} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)})$ time.

6 Parameterizing by $k_r + k_\ell$ and $b + k_r$

In the previous sections we saw that GEN- RBSC- LINES parameterized by r is para-NP-complete and is W[1]-hard parameterized by k_{ℓ} . So there is no hope of an FPT algorithm unless P = NP or FPT =W[1], when parameterized by r and k_{ℓ} respectively. As a consequence, we consider combining different natural parameters with r to see if this helps to find FPT algorithms. In fact, in this section, we describe a FPT algorithm for GEN- RBSC- LINES parameterized by $k_{\ell} + k_r$. Since $k_r \leq r$, this implies that GEN- RBSC- LINES parameterized by $k_{\ell} + r$ is FPT. This is one of our main technical/algorithmic contribution. Also, since $k_{\ell} \leq b$ for any minimal solution family of an instance, it follows that GEN- RBSC- LINES parameterized by $b + k_r$ belongs to FPT. It is natural to ask whether the GEN- RBSC problem, that is, where sets in the family are arbitrary subsets of the universe and need not correspond to lines, is FPT parameterized by $k_{\ell} + k_r$. In fact, Theorem 10.1 states that the problem is W[1]-hard even when each set is of size three and contains at least two red points. This shows that indeed restricting ourselves to sets corresponding to lines makes the problem tractable.

We start by considering a simpler case, where the input instance is such that every line contains exactly 1 blue point. Later we will show how we can reduce our main problem to such instances. By the restrictions assumed on the input, no two blue points can be covered by the same line and any solution family must contain at least *b* lines. Thus, $b \le k_{\ell}$ or else, it is a NO instance. Also, a minimal solution family will contain at most *b* lines as every line covers a blue point not covered by any other line in the solution. Thus a minimal solution family, in this special case, contains exactly *b* lines. Let $G_{\mathcal{F}'}$ be the intersection graph that corresponds to a minimal solution \mathcal{F}' . Recall, that in $G_{\mathcal{F}'}$ vertices correspond to lines in \mathcal{F}' and there is an edge between two vertices in $G_{\mathcal{F}'}$ if the corresponding lines intersect either at a blue point or a red point. Next, we define notions of *good tuple* and *conformity* which will be useful in designing the FPT algorithm for the special case. Essentially, a good tuple provides a numerical representation of connected components of $G_{\mathcal{F}'}$. **Definition 6.1** Given an instance (R, B, \mathcal{F}) of GEN- RBSC- LINES we call a tuple

$$(b, p, s, P, \{I'_1, \ldots, I'_s\}, (k^1_r, k^2_r, \ldots, k^s_r))$$

good if the following hold.

- (a) Integers $p \le k_r$ and $s \le b \le k_\ell$; Here b is the number of blue vertices in the instance.
- (b) $P = P_1 \cup \cdots \cup P_s$ is an *s*-partition of *B*;
- (c) For each $1 \le i \le s$, I'_i is an ordering for the blue points in part P_i ;
- (d) Integers k_r^i , $1 \le i \le s$, are such that $\sum_{1 \le i \le s} k_r^i = p$.

Below, we define the relevance of good tuples in the context of our problem.

Definition 6.2 We say that the minimal solution family \mathcal{F}' conforms with a good tuple $(b, p, s, P, \{I'_1, \ldots, I'_s\}, (k_r^1, k_r^2, \ldots, k_r^s))$ if the following properties hold:

- 1. The components C_1, \ldots, C_s of $G_{\mathcal{F}'}$ give the partition $P = P_1, \ldots, P_s$ on the blue points.
- 2. For each component C_i , $1 \le i \le s$, let $t_i = |P_i|$. Let $I'_i = b^i_1, \ldots, b^i_{t_i}$ be an ordering of blue points in P_i . Furthermore assume that $L^i_j \in \mathcal{F}'$ covers the blue point b^i_j . I'_i has the property that for all $j \le t_i$ $G_{\mathcal{F}'}[\{L^i_1, \ldots, L^i_j\}]$ is connected. In other words for all $j \le t_i$, L^i_j intersects with at least one of the lines from the set $\{L^i_1, \ldots, L^i_{j-1}\}$. Notice that, by minimality of \mathcal{F}' , the point of intersection for such a pair of lines is a red point.
- 3. \mathcal{F}' covers $p \leq k_r$ red points.
- 4. In each component C_i , k_r^i is the number of red points covered by the lines in that component. It follows that $\sum_{1 \le i \le s} k_r^i = p$. In other words, the integers k_r^i are a composition of p.

The next lemma says that the existence of a minimal solution subfamily \mathcal{F}' results in a conforming good tuple.

Lemma 6.1 Let (U, \mathcal{F}) be an input to GEN- RBSC- LINES parameterized by $k_{\ell} + k_r$, such that every line contains exactly 1 blue point. If there exists a solution subfamily \mathcal{F}' then there is a conforming good tuple.

Proof Let \mathcal{F}' be a minimal solution family of size $b \leq k_{\ell}$ that covers $p \leq k_r$ red points. Let $G_{\mathcal{F}'}$ have *s* components, C_1, C_2, \ldots, C_s , where $s \leq k_{\ell}$. For each $i \leq s$, let \mathcal{F}_{C_i} denote the set of lines corresponding to the vertices of C_i . $P_i = B \cap \mathcal{F}_{C_i}$, $t_i = |P_i|$ and $k_r^i = |R \cap \mathcal{F}_{C_i}|$. In this special case and by minimality of $\mathcal{F}', |\mathcal{F}_{C_i}| = t_i$. As C_i is connected, there is a sequence $\{L_1^i, L_2^i, \ldots, L_{t_i}^i\}$ for the lines in \mathcal{F}_{C_i} such that for all $j \leq t_i$ we have that $G_{\mathcal{F}'}[\{L_1^i, \ldots, L_j^i\}]$ is connected. This means that, for all $j \leq t_i L_j^i$ intersects with at least one of the lines from the set $\{L_1^i, \ldots, L_{j-1}^i\}$. By minimality of \mathcal{F}' , the point of intersection for such a pair of lines is a red point. For all $j \leq t_i$, let L_j^i cover the blue point b_j^i . Let $I_i' = b_1^i, b_2^i, \ldots, b_{t_i}^i$. The tuple $(b, p, s, P = P_1 \cup P_2 \ldots \cup P_s, \{I'_1, \ldots, I'_s\}, (k^1_r, k^2_r, \ldots, k^s_r))$ is a good tuple and it also conforms with \mathcal{F}' . This completes the proof.

The idea of the algorithm is to generate all good tuples and then check whether there is a solution subfamily \mathcal{F}' that conforms to it. The next lemma states that we can check for a conforming minimal solution family when we are given a good tuple.

Lemma 6.2 For a good tuple $(b, p, s, P, \{I'_1, \ldots, I'_s\}, (k^1_r, k^2_r, \ldots, k^s_r))$, there is an algorithm running in $\mathcal{O}(b\ell p^b k_r)$ time, that either says that there is no conforming minimal solution family \mathcal{F}' or says that there is a minimal solution family \mathcal{F}' that conforms with a good tuple $(b, \hat{p}, \hat{s}, \hat{P}, \{\hat{I}_1, \hat{I}_2, \ldots, \hat{I}_{\hat{s}}\}, (\hat{k}_r^{-1}, \hat{k}_r^{-2}, \ldots, \hat{k}_r^{-\hat{s}}))$ such that:

- 1. $\hat{p} \le p$,
- 2. $\hat{s} \leq s$,
- 3. The partition P is a refinement of the partition \hat{P} .

Proof The first step of the algorithm is a preprocessing step, where for each line we store the set of red vertices that are covered by the line. After that, the algorithm essentially builds a search tree for each partition P_i , $1 \le i \le s$. For each part P_i , we define a set of points R'_i which is initially an empty set.

For each $1 \le i \le s$, let $t_i = |P_i|$ and let $I'_i = b^i_1, \ldots, b^i_{t_i}$ be the ordering of blue points in P_i . Our objective is to check whether there is a subfamily $\mathcal{F}'_i \subseteq \mathcal{F}$ such that it covers $b^i_1, \ldots, b^i_{t_i}$, and at most k^i_r red point. At any stage of the algorithm, we have a subfamily \mathcal{F}'_i covering b^i_1, \ldots, b^i_j and at most k^i_r red points. In the next step we try to enlarge \mathcal{F}'_i in such a way that it also covers b^i_{j+1} , but still covers at most k^i_r red points and stay connected. In some sense we follow the ordering given by I'_i to build \mathcal{F}'_i .

Initially, $\mathcal{F}'_i = \emptyset$. At any point of the recursive algorithm we represent the problem to be solved by the following tuple: $(\mathcal{F}'_i, R'_i, (b^i_j, \dots, b^i_{t_i}), k^i_r - |R'_i|)$. We start the process by guessing the line in \mathcal{F} that covers b^i_1 , say L^i_1 . That is, for every $L \in \mathcal{F}$ such that b^i_1 is contained in L we recursively check whether there is a solution to the tuple $(\mathcal{F}'_i := \mathcal{F}'_i \cup \{L\}, R'_i := R'_i \cup (R \cap L), (b^i_2, \dots, b^i_{t_i}), k^i_r := k^i_r - |R'_i|)$. If any tuple returns YES then we return that there is a subset $\mathcal{F}'_i \subseteq \mathcal{F}$ which covers $b^i_1, \dots, b^i_{t_i}$, and at most k^i_r red points.

Now suppose we are at an intermediate stage of the algorithm and the tuple we have is $(\mathcal{F}'_i, R'_i, (b^i_j, \ldots, b^i_{l_i}), k^i_r)$. Let \mathcal{L} be the set of lines such that each line contains b^i_j and a red point from R'_i (The second condition is to ensure that the lines form a connected component). Clearly, $|\mathcal{L}| \leq |R'_i| \leq k^i_r$. For every line $L \in \mathcal{L}$, we recursively check whether there is a solution to the tuple $(\mathcal{F}'_i := \mathcal{F}'_i \cup \{L\}, R'_i := R'_i \cup (R \cap L), (b^i_{j+1}, \ldots, b^i_{l_i}), k^i_r := k^i_r - |R'_i|)$. If any tuple returns YES then we return that there is a subset $\mathcal{F}'_i \subseteq \mathcal{F}$ which covers $b^i_1, \ldots, b^i_{l_i}$, and at most k^i_r red points.

Let μ represent the number of uncovered blue points in the algorithm. Note that $\mu \leq t_i$. At each stage, the value of μ drops by one and, except for the first step, the algorithm recursively solves at most $k_r^i \leq p$ subproblems. In each step of the algorithm, a line is chosen and updates are made according to the set of red points covered by that line. In other words, the set of red points that are covered by the line but

not in R'_i before, are included in R'_i . Also, the budget for the remaining red points that may be included in R'_i is modified. The adjustments with respect to the new red points covered by the chosen line can be done in time linear to the size of the set of these red points. As argued earlier, each chosen line has at most k_r points covered by it. Let T()be the running time function for the algorithm once a line is chosen. Recursively, for the choice of one line in the first step, $T(\mu) \leq pT(\mu - 1) + k_r$. Thus, for the choice of one line in the first step, the recursive algorithm runs in time $O(t_i p^{t_i} k_r)$. In the first step, there are ℓ guesses for the line covering b_1 . This implies that for each partition the algorithm takes at most $O(|\mathcal{F}|t_i(p^{t_i}k_r)) = O(\ell t_i p^{t_i}k_r)$ time.

By definition of the good tuple corresponding to the potential minimal solution \mathcal{F}' , it is possible that some of the parts in P will merge to form the set of blue points in a component of \mathcal{F}' . In particular, in our procedure, a red point could be included in more than one set R'_i (the set of covered red points corresponding to a part $P_i \in P$). Moreover, notice that the lines in the input instance are partitioned according to the blue points contained in it. Thus, the search corresponding to each part P_i is independent of those in other parts. In effect, we are searching for the components for $G_{\mathcal{F}'}$ in the input instance, in parallel.

Suppose for each P_i we are successful in finding a minimal set of lines covering exactly the blue points of P_i while covering at most k_r^i red points. Let the union of these minimal sets result in a solution \mathcal{F}' . Since each line in \mathcal{F}' covers a unique blue point, this solution is minimal. This also means that all blue points have been covered, possibly with the merging of some of the parts of P through red points, that at most $p \leq k_r$ red points have been covered and at most $b \leq k_\ell$ lines belong to \mathcal{F}' . Let $(b, \hat{p}, \hat{s}, \hat{P}, \{\hat{I}_1, \hat{I}_2, \dots, \hat{I}_{\hat{s}}\}, (\hat{k}_r^{-1}, \hat{k}_r^{-2}, \dots, \hat{k}_r^{-\hat{s}}))$ be the conforming tuple for \mathcal{F}' as described in the proof of Lemma 6.1. By construction, Pis a refinement of $\hat{P}, \hat{s} \leq s$ and $\hat{p} \leq p$. Thus, we conclude that a solution family \mathcal{F}' that conforms to a good tuple $(b, \hat{p}, \hat{s}, \hat{P}, \{\hat{I}_1, \hat{I}_2, \dots, \hat{I}_{\hat{s}}\}, (\hat{k}_r^{-1}, \hat{k}_r^{-2}, \dots, \hat{k}_r^{-\hat{s}}))$, with the properties mentioned in the Lemma statement, exists and hence the input instance is a YES instance. If the algorithm outputs NO after the exhaustive search, then we conclude that there is no minimal solution family that conforms to the input good tuple $(b, p, s, P, \{I'_1, \dots, I'_s\}, (k_r^{1}, k_r^{2}, \dots, k_r^{s}))$.

The time taken for the described procedure in each part is at most $O(\ell t_i p^{t_i} k_r)$. Hence, the total time taken to check if there is a conforming minimal solution family \mathcal{F}' is at most

$$\mathcal{O}\left(\ell \cdot \sum_{i=1}^{s} t_i p^{t_i} k_r\right) = \mathcal{O}(b\ell p^b k_r).$$

The equation follows from the fact that $\sum_{i=1}^{s} t_i = b$. This concludes the proof. \Box

We are ready to describe our FPT algorithm for this special case of GEN-RBSC-LINES parameterized by $k_{\ell} + k_r$.

Lemma 6.3 Let $(U, \mathcal{F}, k_{\ell}, k_r)$ be an input to GEN- RBSC- LINES such that every line contains exactly 1 blue point. Then we can check whether there is a solution subfamily \mathcal{F}' to this instance in time $k_{\ell}^{\mathcal{O}(k_{\ell})} \cdot k_{r}^{\mathcal{O}(k_{r})} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$ time.

Proof Lemma 6.1 implies that for the algorithm all we need to do is to enumerate all possible good tuples $(b, p, s, P, \{I'_1, \ldots, I'_s\}, (k^1_r, k^2_r, \ldots, k^s_r))$, and for each tuple, check whether there is a conforming minimal solution family. Later, we use the algorithm described in Lemma 6.2.

We first give an upper bound on the number of tuples and how to enumerate them.

- 1. There are at most k_{ℓ} choices for *s* and k_r choices for *p*.
- 2. There can be at most $b^{k_{\ell}}$ choices for *P* which can be enumerated in $\mathcal{O}(b^{k_{\ell}} \cdot k_{\ell})$ time.
- 3. For each $j \le s$, I'_j is ordering for blue points in P_i . Thus, if $|P_i| = t_i$, then the number of ordering tuples $\{I'_1, \ldots, I'_s\}$ is upper bounded by $\prod_{i=1}^s t_i! \le \prod_{i=1}^s t_i^{t_i} \le \prod_{i=1}^s b^{t_i} = b^b$. Such orderings can be enumerated in $\mathcal{O}(b^b)$ time.
- 4. For a fixed $p \le k_r$, $s \le k_\ell$, there are at most $\binom{p+s-1}{s-1}$ solutions for $k_r^1 + k_r^2 + \dots + k_r^s = p$ and this set of solutions can be enumerated in $\mathcal{O}(\binom{p+s-1}{s-1} \cdot ps)$ time. Notice that if $p \ge s$ then the time required for enumeration is $\mathcal{O}((2p)^p \cdot ps)$. Otherwise, the required time is $\mathcal{O}((2s)^s \cdot ps)$. As $p \le k_r$ and $s \le k_\ell$, the time required to enumerate the set of solutions is $\mathcal{O}(k_\ell^{\mathcal{O}(k_\ell)} k_r^{\mathcal{O}(k_r)} \cdot k_\ell k_r)$.

Thus we can generate the set of tuples in time $k_{\ell}^{\mathcal{O}(k_{\ell})} \cdot k_{r}^{\mathcal{O}(k_{r})}$. Using Lemma 6.2, for each tuple we check in at most $\mathcal{O}(k_{r}^{k_{\ell}} \cdot k_{\ell}\ell k_{r})$ time whether a solution family can be constructed. If there is no tuple that leads to a solution family, we know that the input instance is a NO instance. The total time for this algorithm is $k_{\ell}^{\mathcal{O}(k_{\ell})} k_{r}^{\mathcal{O}(k_{\ell})} k_{r}^{\mathcal{O}(k_{\ell})} \cdot$ $(|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$. Again, if $k_{r} \leq k_{l}$ then $k_{r}^{\mathcal{O}(k_{\ell})} = k_{\ell}^{\mathcal{O}(k_{\ell})}$. Otherwise, $k_{r}^{\mathcal{O}(k_{\ell})} = k_{r}^{\mathcal{O}(k_{r})}$. Either way, it is always true that $k_{r}^{\mathcal{O}(k_{\ell})} = k_{\ell}^{\mathcal{O}(k_{\ell})} k_{r}^{\mathcal{O}(k_{r})}$. Thus, we can simply state the running time to be $k_{\ell}^{\mathcal{O}(k_{\ell})} \cdot k_{r}^{\mathcal{O}(k_{r})} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$.

We return to the general problem of GEN- RBSC- LINES parameterized by $k_{\ell} + k_r$. Instances in this problem may have lines containing 2 or more blue points.

We use the results and observations described above to arrive at an FPT algorithm for GEN- RBSC- LINES parameterized by $k_{\ell} + k_r$.

Theorem 6.1 GEN- RBSC- LINES parameterized by $k_{\ell} + k_r$ is FPT, with an algorithm that runs in $k_{\ell}^{\mathcal{O}(k_{\ell})} \cdot k_r^{\mathcal{O}(k_r)} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$ time.

Proof Given an input $(U, \mathcal{F}, k_{\ell}, k_r)$ for GEN-RBSC-LINES parameterized by $k_{\ell} + k_r$, we do some preprocessing to make the instance simpler. We exhaustively apply Reduction Rules 1, 2 and 3. After this, by Observation 2, the reduced equivalent instance has at most $\binom{k_{\ell}}{2}$ blue points if it is a YES instance.

A minimal solution family can be broken down into two parts: the set of lines containing at least 2 blue points, and the remaining set of lines which contain exactly 1 blue point. Let us call these sets \mathcal{F}_2 and \mathcal{F}_1 respectively. We start with the following observation.

Observation 3 Let $\mathcal{F}'' \subseteq \mathcal{F}$ be the set of lines that contain at least 2 blue points. There are at most $\binom{k_l^4}{k_l}$ ways in which a solution family can intersect with \mathcal{F}'' . *Proof* Since $b \leq {\binom{k_{\ell}}{2}}$, it follows from Observation 1 that $|\mathcal{F}''| \leq k_{\ell}^4$. For any solution family, there can be at most k_{ℓ} lines containing at least 2 blue points. Since the number of subsets of \mathcal{F}'' of size at most k_{ℓ} is bounded by $k_{\ell}^{4k_{\ell}}$, the observation is true. \Box

From Observation 3, there are $k_{\ell}^{4k_{\ell}}$ choices for the set of lines in \mathcal{F}_2 . We branch on all these choices of \mathcal{F}_2 . On each branch, we reduce the budget of k_{ℓ} by the number of lines in \mathcal{F}_2 and the budget of k_r by $|R \cap \mathcal{F}_2|$. Also, we make some modifications on the input instance: we delete all other lines containing at least 2 blue points from the input instance. We delete all points of U covered by \mathcal{F}_2 and all lines passing through blue points covered by \mathcal{F}_2 . Our modified input instance in this branch now satisfies the assumption of Lemma 6.3 and we can find out in $k_{\ell}^{\mathcal{O}(k_{\ell})} k_r^{\mathcal{O}(k_r)} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$ time whether there is a minimal solution family \mathcal{F}_1 for this reduced instance. If there is, then $\mathcal{F}_2 \cup \mathcal{F}_1$ is a minimal solution for our original input instance and we correctly say YES. Thus the total running time of this algorithm is $k_{\ell}^{\mathcal{O}(k_{\ell})} \cdot k_r^{\mathcal{O}(k_r)} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$.

It may be noted here that for a special case where we can use any line in the plane as part of the solution, the second part of the algorithm becomes considerably simpler. Here for each blue point b, we can use an arbitrary line containing only b and no red point.

Corollary 6.1 GEN- RBSC- LINES parameterized by $k_{\ell} + d$, where every line contains at most d red points, is FPT. The running time of the FPT algorithm is $(dk_{\ell})^{\mathcal{O}(dk_{\ell})}$. $(|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$. The problem remains FPT for all parameter sets Γ' that contain $\{k_{\ell}, d\}$ or $\{b, d\}$.

Proof In this special case, any solution family can contain at most dk_{ℓ} red points. Hence we can safely assume that $k_r \leq dk_{\ell}$ and apply Theorem 6.1.

6.1 Kernelization for GEN-RBSC-LINES Parameterized by $k_{\ell} + k_r$ and $b + k_r$

We give a polynomial parameter transformation from SET COVER parameterized by universe size *n*, to GEN- RBSC- LINES parameterized by $k_{\ell} + k_r + b$. Proposition 1(ii) implies that on parameterizing by any subset of the parameters { k_{ℓ}, k_r, b }, we will also obtain a negative result for polynomial kernels.

Theorem K.3 GEN- RBSC- LINES parameterized by $k_{\ell} + k_r + b$ does not allow a polynomial kernel unless co-NP \subseteq NP/poly.

Proof Let (U, S) be a given instance of SET COVER. Let |U| = n, |S| = m. We construct an instance $(R \cup B, \mathcal{F})$ of GEN-RBSC-LINES as follows. We assign a blue point $b_u \in B$ for each element $u \in U$ and a red point $r_S \in R$ for each set $S \in S$. The red and blue points are placed such that no three points are collinear. We add a line between b_u and r_S if $u \in S$ in the SET COVER instance. Thus the GEN-RBSC-LINES instance $(R \cup B, \mathcal{F})$ that we have constructed has b = n, r = m and $\ell = \sum_{S \in S} |S|$. We set $k_r = k$ and $k_\ell = n$.

Claim All the elements in (U, S) can be covered by *k* sets if and only if there exist *n* lines in $(R \cup B, F)$ that contain all blue points but only *k* red points.

Proof Suppose (U, S) has a solution of size k, say $\{S_1, S_2, \dots, S_k\}$. The red points in the solution family for GEN- RBSC- LINES are $\{r_{S_1}, r_{S_2}, \dots, r_{S_k}\}$ corresponding to $\{S_1, S_2, \dots, S_k\}$. For each element $u \in U$, we arbitrarily assign a covering set S_u from $\{S_1, S_2, \dots, S_k\}$. The solution family is the set of lines defined by the pairs $\{(b_u, r_{S_u}) \mid u \in U\}$. This covers all blue points.

Conversely, if $(R \cup B, \mathcal{F})$ has a solution family \mathcal{F}' covering *k* red points and using at most *n* lines, the sets in \mathcal{S} corresponding to the red points in \mathcal{F}' cover all the elements in (U, \mathcal{S}) .

If k > n, then the SET COVER instance is a trivial YES instance. Hence, we can always assume that $k \le n$. This completes the proof that GEN- RBSC- LINES parameterized by $k_{\ell} + k_r + b$ cannot have a polynomial sized kernel unless co-NP \subseteq NP/poly.

7 Hyperplanes: Parameterized by $k_{\ell} + k_r$

Theorem 7.1 GEN- RBSC for hyperplanes in \mathbb{R}^d , for a fixed positive integer d, is W[1]-hard when parameterized by $k_{\ell} + k_r$.

Proof The proof of hardness follows from a reduction from k-CLIQUE problem. The proof follows a framework given in [18].

Let (G(V, E), k) be an instance of k-CLIQUE problem. Our construction consists of a $k \times k$ matrix of gadgets G_{ij} , $1 \le i, j, \le k$. Consecutive gadgets in a row are connected by horizontal connectors and consecutive gadgets in a column are connected by vertical connectors. Let us denote the horizontal connector connecting the gadgets G_{ij} and G_{ih} as $H_{i(jh)}$ and the vertical connector connecting the gadgets G_{ij} and G_{hj} as $V_{(ih)j}$, $1 \le i, j, h \le k$.

Gadgets: The gadget G_{ij} contains a blue point b_{ij} and a set R_{ij} of d - 2 red points. In addition there are n^2 sets $R'_{ij}(a, b)$, $1 \le a, b \le n$, each having two red points each.

Connectors: The horizontal connector $H_{i(jh)}$ has a blue point $b_{i(jh)}$ and a set $R_{i(jh)}$ of d - 2 red points. Similarly, the vertical connector $V_{(ih)j}$ a blue point $b_{(ih)j}$ and a set $R_{(ih)j}$ of d - 2 red points.

The points are arranged in general position i.e., no set of d + 2 points lie on the same d-dimensional hyperplane. In other words, any set of d + 1 points define a distinct hyperplane.

Hyperplanes: Assume $1 \le i, j, h \le k$ and $1 \le a, b, c \le n$. Let $P_{ij}(a, b)$ be the hyperplane defined by the d + 1 points of $b_{ij} \cup R_{ij} \cup R'_{ij}(a, b)$. Let $P^h_{i(jh)}(a, b, c)$ be the hyperplane defined by d + 1 points of $b_{i(jh)} \cup R_{i(jh)} \cup r_1 \cup r_2$ where $r_1 \in R'_{ij}(a, b)$ and $r_2 \in R'_{ih}(a, c)$. Let $P^v_{(ij)h}(a, b, c)$ be the hyperplane defined by d + 1 points of $b_{i(jh)} \cup R_{i(jh)} \cup r_1 \cup r_2$ where $r_1 \in R'_{ij}(a, b)$ and $r_2 \in R'_{ih}(a, c)$. Let $P^v_{(ij)h}(a, b, c)$ be the hyperplane defined by d + 1 points of $b_{(ij)h} \cup R_{(ij)h} \cup r_1 \cup r_2$ where $r_1 \in R'_{ih}(a, c)$ and $r_2 \in R'_{ih}(b, c)$.

For each edge $(a, b) \in E(G)$, we add k(k-1) hyperplanes of the type $P_{ij}(a, b), i \neq j$. Further, for all $1 \le a \le n$, we add k hyperplanes of the type $P_{ii}(a, a), 1 \le i \le k$. The hyperplane $P_{i(jh)}^h(a, b, c)$ containing the blue point $b_{i(jh)}$ in a horizontal connector, is added to the construction if $P_{ij}(a, b)$ and $P_{ih}(a, c)$ are present in the construction. Similarly, the hyperplane $P_{(ij)h}^{v}(a, b, c)$ containing the blue point $b_{(ij)h}$ in a vertical connector, is added to the construction if $P_{ih}(a, c)$ and $P_{jh}(b, c)$ are present in the construction.

Thus our construction has $k^2 + 2k(k-1)$ blue points, $(k^2 + 2k(k-1))(d-2) + 2n^2k^2$ red points and $O((m^2k^2))$ hyperplanes.

Claim G has a *k*-clique if and only if all the blue points in the constructed instance can be covered by $k^2 + 2k(k-1)$ hyperplanes covering at most $k^2d + 2k(k-1)(d-2)$ red points.

Proof Assume *G* has a clique of size *k* and let $\{a_1, a_2, \dots, a_k\}$ be the vertices of the clique. Now we show a set cover of the desired size exists. Choose *k* hyperplanes, $P_{ii}(a_i, a_i), 1 \le i \le k$, to cover the diagonal gadgets. To cover other gadgets, G_{ij} , choose the hyperplanes $P_{ij}(a_ia_j)$ and to cover the connectors, $H_{i(jh)}$ and $V_{(ih)j}$, choose the hyperplanes $P_{ij}^h(a_i, a_j, a_h)$ and $P_{(ij)h}^v(a_i, a_j, a_h)$. The fact that $\{a_1, a_2, \dots, a_k\}$ forms a clique implies that these hyperplanes do exist in the construction.

Now assume a set cover of the given size exists. To cover the blue point b_{ij} in the gadget G_{ij} , any hyperplane adds d red points. Also to cover the blue point in each connector, we need to add d - 2 extra red points. Since each hyperplane contains d red points and we have already used up our budget of red points, each hyperplane covering the connector points should reuse two red points that have been used in covering gadgets. By construction, this is possible only when all gadgets in a row(column) are covered by hyperplanes corresponding to edges incident on the same vertex viz. the vertex corresponding to the hyperplane covering the diagonal gadget in the row(column). This implies that G has a clique.

8 Multivariate Complexity of GEN-RBSC-LINES: Proof of Theorem 1.1

The first part of Theorem 1.1 (parameterized complexity dichotomy) follows from Theorems 3.1, K.1, 5.1 and 6.1. Recall that $\Gamma = \{\ell, r, b, k_{\ell}, k_r\}$. To show the kernelization dichotomy of the parameterizations of GEN- RBSC- LINES that admit FPT kernels we proceed as follows:

- Show that the problem admits a polynomial kernel parameterized by ℓ (Theorem K.1). This implies that for all Γ' that contains ℓ , the parameterization admits a polynomial kernel.
- Show that the problem does not admit a polynomial kernel when parameterized by $k_{\ell} + k_r + b$ (Theorem K.3). This implies that for all subsets of $\{k_{\ell}, k_r, b\}$, the parameterization does not allow a polynomial kernel.
- The remaining FPT variants of GEN- RBSC- LINES correspond to parameter sets Γ' that contain either r or $\{r, b\}$ together. Recall that, $k_r \leq r$ and $k_\ell \leq b$. The two smallest combined parameters for which we can not infer the kernelization complexity from Theorem K.3 are $r + k_\ell$ and r + b. We show below (Theorem K.4) that GEN- RBSC admits a quadratic kernel parameterized by $r + k_\ell$. Since in any minimal solution family $k_\ell \leq b$, this also implies a quadratic kernel for the

parameterization r + b. Thus, if parameterization by a set Γ' , which contains either r or $\{r, b\}$, allows an FPT algorithm then it also allows a polynomial kernel.

Theorem K.4 GEN- RBSC- LINES parameterized by $k_{\ell} + r$ admits a polynomial kernel.

Proof Given an instance of GEN- RBSC- LINES we first exhaustively apply Reduction Rules 1, 2 and 3 and obtain an equivalent instance. By Observation 2, the reduced instance has at most $b \le k_{\ell}^2$ blue points. By Observation 1, the number of lines containing at least two points is $\binom{r+b}{2}$. After applying Reduction Rule 1, there are no lines with only one red point. Also, for a blue point b_i , if there are many lines that contain only b_i , then we can delete all but one of those lines. Therefore, the number of lines that contain exactly one point is bounded by b. Thus, we get a kernel of k_{ℓ}^2 blue points, $\binom{r+k_{\ell}^2}{2} + k_{\ell}^2$ lines and r red points. This concludes the proof.

Combining Theorems K.1, K.3 and K.4 and the discussion above we prove the second part of Theorem 1.1 (kernelization dichotomy).

9 Parameterized Landscape for RED BLUE SET COVER WITH LINES

Until now our main focus was the GEN- RBSC- LINES problem. In this section, we study the original RBSC- LINES problem. Recall that the original RBSC- LINES problem differs from the GEN- RBSC- LINES problem in the following way—here our objective is *only* to minimize the number of red points that are contained in a solution subfamily, and *not* the size of the subfamily itself. That is, $k_{\ell} = |\mathcal{F}|$. This change results in a slightly different landscape for RBSC- LINES compared to GEN- RBSC- LINES. As before let $\Gamma = \{\ell, r, b, k_{\ell}, k_r\}$. We first observe that for all those $\Gamma' \subseteq \Gamma$ that do not contain k_{ℓ} as a parameter and GEN- RBSC- LINES is FPT parameterized by Γ' . RBSC- LINES is also FPT parameterized by Γ' . Next we list out the subsets of parameters for which the results do not follow from the result on GEN- RBSC- LINES.

- RBSC- LINES becomes FPT parameterized by r.
- W[2]-hard parameterized by k_r .

9.1 RBSC-LINES Parameterized by r

Theorem K.5 RBSC- LINES parameterized by r is FPT. Furthermore, RBSC- LINES parameterized by r does not allow a polynomial kernel unless co-NP \subseteq NP/poly.

Proof We proceed by enumerating all possible k_r -sized subsets of R. For each subset, we can check in polynomial time whether the lines spanned by exactly those points cover all blue points. This is our FPT algorithm, which runs in $\mathcal{O}(2^r \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)})$.

Using Proposition 1, it is enough to show a polynomial parameter transformation from SET COVER parameterized by size m of the set family, to RBSC-LINES parameterized by r. The reduction is exactly the same as the one given in the proof of Theorem K.3. This gives the desired second part of the theorem.

9.2 RBSC-LINES Parameterized by k_r

In this section we study parameterization by k_r and some special cases which leads to FPT algorithm. We prove that RBSC-LINES parameterized by k_r is W[2]-hard. From Proposition 1, SET COVER parameterized by solution family size k is W[2]-hard. The W[2]-hardness of RBSC-LINES parameterized by k_r can be proved by a many-one reduction from SET COVER parameterized by k. The reduction is exactly the one that is given in Theorem K.3.

Theorem 9.1 RBSC- LINES parameterized by k_r is W[2]-hard.

9.2.1 FPT Result Under Special Assumptions

In this section we consider a special case, where in the given instance every line contains either no red points or at least 2 red points. There are two reasons motivating the study of this special case. Firstly, in the W[2]-hardness proof we crucially used the fact that the constructed RBSC-LINES instance has a set of lines with exactly 1 red point. Thus, it is necessary to check if this is the reason leading to the hardness of the problem. Secondly, if we look at RBSC (sets in the family can be arbitrary) parameterized by k_r and assumed that in the given instance every line contains either no red points or at least 2 red points, even then the problem is W[1]-hard (see Theorem 10.1). However, when we consider RBSC-LINES parameterized by k_r and where in the given instance every set contains either no red points or at least 2 red points, the problem is FPT.

For our algorithm we also need the following new reduction rule.

Reduction Rule 4 If there is a set $S \in \mathcal{F}$ with only blue points then delete that set from \mathcal{F} and include the set in the solution.

Lemma 9.1 Reduction Rule 4 is safe.

Proof Since the parameter is k_r , there is no size restriction on the number of lines in the solution subfamily \mathcal{F}' . If \mathcal{F}' is a solution subfamily and $S \in \mathcal{F}$ then under this parameterization, $\mathcal{F}' \cup \{S\}$ is also a solution family covering all blue points and at most k_r red points. This shows that Reduction Rule 4 is valid.

Theorem 9.2 RBSC- LINES parameterized by k_r , where the input instance has every set containing at least 2 red points or no red points at all, has an algorithm with running time $k_r^{\mathcal{O}(k_r^2)} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$.

Proof Given an instance of RBSC-LINES, we first exhaustively apply Reduction Rules 1, 2 and 4 and obtain an equivalent instance. At the end of these reductions we obtain an equivalent instance where every line has at least 1 blue point and at least 2 red points, but at most k_r red points.

Suppose \mathcal{F}' is a solution family. Since a line with a red point has at least 2 red points, by Observation 1, the total number of sets that can contain the red points covered by \mathcal{F}' is at most $\binom{k_r}{2}$. This means that, if the input instance is a YES instance, there exists

a solution family with at most $k_{\ell} = {\binom{k_r}{2}}$ lines. Now we can apply the algorithm for GEN- RBSC- LINES parameterized by $k_{\ell} + k_r$ described in Theorem 6.1 to obtain an algorithm for RBSC- LINES parameterized by k_r .

Theorem 9.2 gives an FPT algorithm for RBSC- LINES parameterized by k_r . In what follows we show that the same parameterization does not yield a polynomial kernel for this special case of RBSC- LINES. Towards this we give a polynomial parameter transformation from SET COVER parameterized by universe size *n*, to RBSC- LINES parameterized by k_r and under the assumption that all sets in the input instance have at least 2 red points.

Theorem K.6 RBSC- LINES parameterized by k_r , and under the assumption that all lines in the input have at least 2 red points, does not allow a polynomial kernel unless $co-NP \subseteq NP/poly$.

Proof Let (U, S) be a given instance of the SET COVER problem. We construct an instance $(R \cup B, \mathcal{F})$ of RBSC- LINES as follows. We assign a blue point $b_u \in B$ for each element $u \in U$ and a red point $r_S \in R$ for each set $S \in S$. The red and blue points are placed such that no three points are collinear. We add a line between b_u and r_S if $u \in S$ in the SET COVER instance. To every line *L*, defined by a blue point b_u and a red points r_S , we add a unique red point $r_L \in R$. Thus the RBSC- LINES instance $(R \cup B, \mathcal{F})$ that we have constructed has *n* blue points, $\sum_{s \in S} |S|$ lines and $m + \sum_{s \in S} |S|$ red points. We set $k_r = k + n$.

Claim All the elements in (U, S) can be covered by *k* sets if and only if there exist lines in $(R \cup B, F)$ that contain all blue points but only k + n red points.

Proof Suppose (U, S) has a solution of size k, say $\{S_1, S_2, \ldots, S_k\}$. To each element $u \in U$, we arbitrarily associate a covering set S_u from $\{S_1, S_2, \ldots, S_k\}$. Our solution family \mathcal{F}' of lines are the lines defined by the pairs of points $\{(b_u, r_{S_u}) \mid u \in U\}$. These lines cover all blue points. The number of red points contained in these lines are the *k* red points $\{r_{S_1}, r_{S_2}, \ldots, r_{S_k}\}$ associated with $\{S_1, S_2, \ldots, S_k\}$, and the *n* red points $\{r_L \mid L \in \mathcal{F}'\}$. Therefore, in total there are k + n red points in the solution.

Conversely, suppose $(R \cup B, \mathcal{F})$ has a family \mathcal{F}' covering all blue points and at most k + n red points. The construction ensures that at least n lines are required to cover the n blue points. This also implies that the unique red points belonging to each of these lines add to the number of red points contained in the solution family. The remaining k red points, that are contained in the solution family, correspond to sets in S that cover all the elements in (U, S).

If k > n, then the SET COVER instance is a trivial YES instance. Hence, we can always assume that $k \le n$. This completes the proof that RBSC- LINES parameterized by k_r , and under the assumption that every line in the input instance has at least 2 red points, cannot have a polynomial sized kernel unless co-NP \subseteq NP/poly.

9.3 Proof of Theorem 1.2

Proof of Theorem 1.2 follows from Theorems 1.1, K.5 and 9.1.

10 GENERALIZED RED BLUE SET COVER

In this section we show that for several parameterizations, under which GEN- RBSC-LINES is FPT, the GEN- RBSC problem is not. In this section we give the following three results which complement the corresponding results in the geometric setting.

- 1. GEN- RBSC is W[1]-hard parameterized by $k_{\ell} + k_r$ when every set has size at most three and contains at least two red elements.
- 2. GEN- RBSC is W[2]-hard parameterized by $k_{\ell} + r$ when every set contains at most one red element.
- 3. GEN- RBSC is FPT, parameterized by k_{ℓ} and d, when every set has at most one red element. Here, d is the size of the maximum cardinality set in \mathcal{F} .

10.1 GEN-RBSC Parameterized by $k_{\ell} + k_r$ and $k_{\ell} + r$

Theorem 10.1 GEN- RBSC is W-hard in the following cases:

- (i) When every set contains at least two red elements but at most three elements, and the parameters are $\{k_{\ell}, k_r\}$, the problem is W[1]-hard.
- (ii) When every set contains at most one red element and the parameters are $\{k_{\ell}, r\}$, then the problem is W[2]-hard.

Proof We start by proving the first result. From an instance (G = (V, E), k) of MUL-TICOLORED CLIQUE parameterized by k, we construct an instance $(U = (R, B), \mathcal{F})$ of GEN- RBSC parameterized by $k_{\ell} + k_r$ with the restriction that the size of each set is at most three and there are at least 2 red elements. The construction is as follows.

- Let the given vertex set be $V = V_1 \uplus V_2 \uplus \cdots \uplus V_k$. For every pair $(i, j), 1 \le i < j \le k$, we introduce a new blue element $b_{ij} \in B$. Thus we have $\binom{k}{2}$ blue elements.
- For each vertex $v \in V$ we introduce a new red element $r_v \in R$.
- $U = R \uplus B.$
- For each $e = (u, v) \in E$ such that $u \in V_i, v \in V_j$ and i < j, we define a set $S_e \in \mathcal{F}$ which contains the elements $\{b_{ij}, r_u, r_v\}$.
- We set $k_r = k$ and $k_\ell = \binom{k}{2}$.

This completes our construction. Notice that every set in \mathcal{F} has at least 2 red elements and has size exactly three.

First, assume that (G, k) is a YES instance. Then there is a *k*-sized multi-colored clique *C* in *G*. Let E(C) denote the set of edges of *C*. Pick the subfamily $\mathcal{F}' = \{S_e \mid e \in E(C)\}$ of size $\binom{k}{2}$. Since *C* is a multi-colored clique, for all $(i, j), 1 \leq i < j \leq k$ there is an edge $e_{ij} \in E(C)$ whose endpoints belong to V_i and V_j . Consequently, there is a set $S_{e_{ij}} \in \mathcal{F}'$ that contains b_{ij} . The total number of red elements contained in \mathcal{F}' is equal to the size |V(C)| = k. This shows that (U, \mathcal{F}, k) is a YES instance of GEN-RBSC.

Conversely, suppose (U, \mathcal{F}) is a YES instance of GEN-RBSC. Let \mathcal{F}' be a minimal subfamily of at most $\binom{k}{2}$ sets that covers at most k red elements. Let C be the vertices in G corresponding to the red elements in \mathcal{F}' . Notice that there are $\binom{k}{2}$ blue elements,

no two of which can be covered by the same set. Thus, for all (i, j), $1 \le i < j \le k$, \mathcal{F}' must contain exactly one set $S_e = \{b_{ij}, r_1^{ij}, r_2^{ij}\}$. This implies that for every i, $1 \le i \le k$ the sets in \mathcal{F}' must contain a red element corresponding to a vertex in V_i . Hence, for all $i, 1 \le i \le k$, $C \cap V_i \ne \emptyset$. Also, C forms a clique since the set $S_e = \{b_{ij}, r_1^{ij}, r_2^{ij}\}$ corresponds to the edge between the vertices selected from V_i and V_j . Therefore, (G, k) is a YES instance of MULTICOLORED CLIQUE. This proves that GEN-RBSC, parameterized by $k_\ell + k_r$, is W[1]-hard under the said assumption.

For the second part of the statement, observe that SET COVER is a special case of this problem and therefore, the problem is W[2]-hard.

10.2 A Special Case of GEN-RBSC Parameterized by k_{ℓ}

In this section, we restrict the input instances to those where every set has at most 1 red element and at most *d* blue elements. We design an FPT algorithm for this special case of GEN- RBSC parameterized by $k_{\ell} + d$. It is reasonable to assume that there is *no set* in the given instance with only red elements, since Reduction Rule 1 can be applied to obtain an equivalent instance of GEN- RBSC, under the parameters of $\{k_{\ell}, d\}$.

We were able to show that this problem has an FPT algorithm. However, it was pointed out to us by an anonymous reviewer that there is a simple algorithm based on Dynamic Programming technique. Thus, we present the simpler algorithm.

10.2.1 A Dynamic Programming Algorithm

We give a Dynamic Programming algorithm to solve GEN- RBSC parameterized by $k_l + d$, for the case when all sets contain at most 1 red element and at most d blue elements. Our algorithm guesses the red point that can be added to the solution one by one and also guesses the sets that can cover it and covers the remaining blue points optimally.

Lemma 10.1 There exists a FPT algorithm that solves GEN-RBSC when each set in the input instance contains at most 1 red element and at most d blue elements. The running time of this algorithm is $O(2^{2dk_l}(|U| + |\mathcal{F}|)^{\mathcal{O}(1)})$.

Proof Let $B' \subseteq B$, $r' \in R \cup nil$, $j \in \mathbb{N}$. Let W[B', r'] represent the minimum cardinality of a family $\mathcal{F}' \subseteq \mathcal{F}$ that covers all elements in B' and does not cover any red element except r' (no red element if r' is nil). The value of W[B', r'] is $+\infty$ if no such $\mathcal{F}' \subseteq \mathcal{F}$ exists. Let T[B', j] represent the minimum cardinality of a family $\mathcal{F}' \subseteq \mathcal{F}$ that covers all elements in B' and covers at most j red elements. Clearly the instance is a YES instance if and only if $T[B, k_r] \leq k_l$.

We can compute the value of $T[B, k_r]$ using the following recurrence. T[B', 0] = W[B', nil] $T[B', j] = \min_{r' \in (R \cup nil)} \min_{B'' \subseteq B'} (W[B'', r'] + T[B' \setminus B'', j - 1])$

Similarly we can compute the value of W[B', r'] using the following recurrence. $W[\emptyset, r'] = 0$ $W[B', r'] = 1 + \min_{S \in \mathcal{F}, S \cap R = \emptyset \text{ or } S \cap R = \{r'\}, S \cap B' \neq \emptyset} W[B' \setminus S, r']$

Let us first show that the recurrence for W is correct. The proof is by induction on |B|. When |B| = 0 the recurrence correctly returns zero. When |B| > 0, $W[B' \setminus S, r']$ returns the minimum cardinality of a family $\mathcal{F}' \subseteq \mathcal{F}$ that covers all elements in $B' \setminus \mathcal{F}$ and does not cover any red element except r' (By induction hypothesis). Therefore, $S \cup \mathcal{F}'$ covers all elements in B' and does not cover any red element except r'. Since we are doing this for every $S \in \mathcal{F}$ and take the minimum value, the recurrence indeed returns the minimum cardinality of a family $\mathcal{F}' \subseteq \mathcal{F}$ that covers all elements in B' and does not cover any red element except r'.

Now we show that the recurrence for *T* is correct by induction on *j*. When j = 0, the recurrence returns the value of W[B, nil] which returns the minimum cardinality of a family $\mathcal{F}' \subseteq \mathcal{F}$ that covers all elements in *B'* and does not cover any red element. When j > 0, we consider a number of sets containing the same red element r', paying for the blue elements $B'' \subseteq B'$ they cover, and cover the remaining blue elements $B' \setminus B''$ optimally by induction hypothesis. Since we do this for all red points and return the minimum value, the recurrence is correct.

Running time: To compute the value of $T[B, k_r]$ using the above recurrence, we have to compute at most $2^{|B|}|U|$ values of W and T, which is at most $2^{dk_l}|U|$ in YES-instances. Every value of W can be computed in O(|U|) time using previously computed values. To compute a value of T, we take the minimum over all choices of r' in R, over at most $2^{|B|} \le 2^{dk_l}$ choices of B'', and look up earlier values. Thus the running time is bounded by $O(2^{2dk_l}(|U| + |\mathcal{F}|)^{\mathcal{O}(1)})$.

When it comes to kernelization for this special case, we show that even for GEN-RBSC-LINES parameterized by $k_{\ell} + d$ there cannot be a polynomial kernel unless co-NP \subseteq NP/poly. For this we will give a polynomial parameter transformation from SET COVER parameterized by universe size *n*. The ppt reduction is exactly the one given in Theorem K.3.

Theorem K.7 GEN- RBSC- LINES parameterized by $k_{\ell} + d$, and where every line has at most 1 red element and at most d blue elements, does not allow a polynomial kernel unless co-NP \subseteq NP/poly.

11 Conclusion

In this paper, we provided a complete parameterized and kernelization dichotomy of the GEN- RBSC- LINES and RBSC- LINES problems, under all possible combinations of relevant natural parameters. The results are captured in Table 1. We also studied GEN- RBSC under different parameterizations. The next natural step seems to be a study of the GEN- RBSC problem, when the sets are hyperplanes. Another interesting variant is when the set system has bounded intersection.

We believe that the running time of the FPT algorithm for GEN- RBSC- LINES parameterized by k_{ℓ} , k_r is tight, up to the constants appearing in the exponents. It would be interesting to show that the problems cannot have algorithms with running time dependence on parameters as $k_{\ell}^{o(k_{\ell})} \cdot k_r^{O(k_r)}$ or $k_{\ell}^{O(k_{\ell})} \cdot k_r^{o(k_r)}$, under standard complexity theoretic assumptions (like the Exponential Time Hypothesis).

Parameters	GEN- RBSC- LINES	RBSC- LINES
l	FPT, polynomial kernel	FPT, polynomial kernel
r	para-NP-complete	FPT, no polynomial kernel
b	W[1]-hard	W[1]-hard
k_ℓ	W[1]-hard	_
k _r	para-NP-complete	W[2]-hard
$\ell + r$	FPT, polynomial kernel	FPT, polynomial kernel
$\ell + b$	FPT, polynomial kernel	FPT, polynomial kernel
$\ell + k_{\ell}$	FPT, polynomial kernel	-
$\ell + k_r$	FPT, polynomial kernel	FPT, polynomial kernel
r + b	FPT, polynomial kernel	FPT, polynomial kernel
$r + k_{\ell}$	FPT, polynomial kernel	-
$r + k_r$	para-NP-hard	FPT, no polynomial kernel
$b + k_{\ell}$	W[1]-hard	_
$b + k_r$	FPT, no polynomial kernel	FPT, no polynomial kernel
$k_{\ell} + k_r$	FPT, no polynomial kernel	-
$\ell + r + b$	FPT, polynomial kernel	FPT, polynomial kernel
$\ell + r + k_{\ell}$	FPT, polynomial kernel	_
$\ell + r + k_r$	FPT, polynomial kernel	FPT, polynomial kernel
$\ell + b + k_{\ell}$	FPT, polynomial kernel	_
$\ell + b + k_r$	FPT, polynomial kernel	FPT, polynomial kernel
$\ell + k_{\ell} + k_r$	FPT, polynomial kernel	_
$r + b + k_\ell$	FPT, polynomial kernel	_
$r + b + k_r$	FPT, polynomial kernel	FPT, polynomial kernel
$r + k_{\ell} + k_r$	FPT, polynomial kernel	_
$b + k_{\ell} + k_r$	FPT, no polynomial kernel	_
$\ell + r + b + k_{\ell}$	FPT, polynomial kernel	_
$\ell + r + b + k_r$	FPT, polynomial kernel	FPT, polynomial kernel
$\ell + r + k_{\ell} + k_r$	FPT, polynomial kernel	-
$\ell + b + k_{\ell} + k_r$	FPT, polynomial kernel	-
$r + b + k_{\ell} + k_r$	FPT, polynomial kernel	-
$\ell + r + b + k_{\ell} + k_r$	FPT, polynomial kernel	-

Table 1A summary of our algorithmic results

The kernelization results hold unless co-NP \subseteq NP/poly

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