Recursion Theorems problems

1. Prove that there exists $x_0 \in N$ such that for all y,
$f_{x_0}(y)$ is y^2 if y is even, and $f_{x_0+1}(y)$ otherwise.

   Soln: First, there exists a partial recursive function g in two variables such that $g(x, y) = y^2$ if y is even, and $f_{x+1}(y)$ otherwise.

   Let M be a TM that does the following on input x, y: check if y is even; if so write y^2 on the tape and halt; otherwise simulate the TM with index $x + 1$ on input y. Clearly M computes $g(x, y)$.

   Apply S_{mn} theorem: there exists a total recursive function $\sigma : N \to N$ such that for all y, $f_{\sigma(x)}(y) = g(x, y)$. As σ is total recursive, apply Recursion theorem: σ has a fixed point x_0. We therefore have for all y,
$f_{x_0}(y) = f_{\sigma(x_0)}(y) = g(x_0, y)$
$= y^2$ if y is even and $f_{x_0+1}(y)$ otherwise.

2. Define any fixed point for the total recursive function $\sigma : N \to N$ defined as follows: for $x \in N$, the TM with description $\sigma(x)$ computes the function $f_{\sigma(x)}(y)$ which is 1 if $y = 0$ and $f_x(y + 1)$ otherwise.
Describe a fixed point for σ.

   Soln: Let M be a TM that on input $y \in N$ outputs 1 if $y = 0$ and outputs a constant $a \in N$ otherwise. Let $\hat{x}$ be the index of M. It must be the case that $\hat{x}$ is a fixed point: When $y = 0$, we have $f_{\hat{x}}(y) = 1 = f_{\sigma(\hat{x})}(y)$.
Otherwise, we have $f_{\hat{x}}(y) = a = f_{\hat{x}}(y + 1) = f_{\sigma(\hat{x})}(y)$.

3. Let $\sigma : N \to N$ be any total recursive function. Prove that σ has infinitely many fixed points i.e., there are infinitely many $w \in N$ such that $f_w(y) = f_{\sigma(w)}(y)$ for all y.

   Soln: Suppose there exists a total recursive function σ with finitely many fixed points. Let the set of fixed points be denoted by F. Let g be a partial recursive function such that the indices of all TMs computing g are outside of F (TMs in F compute finitely many partial recursive functions, whereas there are infinitely many partial recursive functions. So such a g exists). That is for all TMs M computing g, if x_M is the index of M then $x_M \notin F$. In other words, for all TMs M computing g, $f_{x_M} \neq f_w$ for every $w \in F$. Let u be an index of some TM computing g. Now, consider a function

$\tau : N \to N$ so that $\tau(x)$ is u if $x \in F$, and $\sigma(x)$ otherwise.
Observe that τ is total recursive:
(i) For any $x \in N$, check whether $x \in F$. This can be done in finite time since F is a finite set.
(ii) If $x \in F$, then set $\tau(x) = u$; otherwise compute $\sigma(x)$ (which is total recursive) and assign the resulting value to $\tau(x)$.
We now argue that τ has no fixed point. If $x \in F$, then $\tau(x) = u$ and since $f_u \neq f_w$ for every $w \in F$, we have (in particular) $f_{\tau(x)} \neq f_x$. Now suppose $x \notin F$. Then by definition of fixed points of $\sigma(x)$, $f_{\tau(x)} = f_{\sigma(x)} \neq f_x$.
Combining the two, we have $f_{\tau(x)} \neq f_x$ for every $x \in N$, thus implying that τ has no fixed points. This contradicts the Recursion theorem. Hence, any total recursive function must have infinitely many fixed points.