Practice problems: PH

1. Show that for every $i \in N$, $\Sigma_i^p = \bigcup_c \Sigma_i TIME(n^c)$. A similar statement can be made for Π_i^p and $\bigcup_c \Pi_i TIME(n^c)$.

2. Show that if $3SAT$ is polynomial time reducible to $\overline{3SAT}$ then $PH = NP$.

3. A DNF formula is a disjunction of conjunctive clauses. Eg: $(x_1 \wedge x_2) \vee (x_3 \wedge x_4 \wedge x_5)$. In the $SUCCINCT - SETCOVER$ problem we are given a collection $S = \{\phi_1, \phi_2, \ldots, \phi_m\}$ of 3-DNF formulae on n variables, and an integer k. The aim is to determine whether there is a subset $S' \subseteq \{1, 2, \ldots, m\}$ of size at most k for which $\vee_{i \in S'} \phi_i$ is a tautology. Show that $SUCCINCT - SETCOVER \in \Sigma_2^p$.

Solutions in the next page. Please try the problems first.

1. We see how to show $\Sigma_i Time(n^c) \in PH$ for each c. This will give RHS $\subseteq$ LHS. A language L is said to be $\Sigma_i TIME(n^c)$ when there is a polynomial time deterministic TM M and constants $c_1, c_2, \dots c_i$ such that $x \in L$ if and only if $\exists u_1 \in 0, 1^{c_1|x|^c} \forall u_2 \in 0, 1^{c_2|x|^c} \dots Q_i u_i \in 0, 1^{c_i|x|^c} M(x, u_1, \dots, u_i) = 1$. If we can get a polynomial q and the u_i's to be from $\{0,1\}^{q(|x|)}$ then we will have given the certificate definition of PH. We can assume that the constants $c, c_1, c_2, \dots, c_i$ are all hardwired in the TM M. Let $q(|x|) = \max_i\{c_1, \dots, c_i\}|x|^c$. Suppose we pad each u_i by $q(|x|) - c_i|x|^c$ many 1's. The description of M will be such that it first reads the first $q(|x|)$ bits and from there only considers the first $c_1|x|^c$ bits to be u_1, and so on. Thus, we obtain a certificate definition of PH for L and we are done.
   The other direction of the proof, i.e., showing $LHS \subseteq RHS$ is easy to show. Please try it using the definition of $\Sigma_i TIME$ classes and the certificate definition of PH.

2. If $3SAT$ is polynomial time reducible to $\overline{3SAT}$ then $NP = coNP$ as $3SAT$ is NP-complete and its complement is coNP-complete.Given that $NP = coNP$ we show that $\Sigma_i^p \subseteq NP$ and $\Pi_i^p \subseteq coNP = NP$. This will result in PH collapsing to NP.
   We can show this by induction on i. In the base case, Consider Σ_2^p. For a language $L \in \Sigma_2^p$, there is a deterministic TM M and a polynomial q such that $x \in L$ if and only if $\exists u_1 \in \{0,1\}^{q(|x|)} \forall u_2 \in \{0,1\}^{q(|x|)} M(x, u_1, u_2) = 1$. Note that when we consider the problem $\forall u_2 \in \{0,1\}^{q(|x|)} M(x, u_1, u_2) = 1$ on inputs (x, u_1), this can be thought of as a problem in $\Pi_1 = coNP = NP$. Thus, there is a deterministic TM M' and a polynomial q' such that $x \in L$ iff $\exists u_1 \in \{0,1\}^{q(|x|)} \exists v_1 \in \{0,1\}^{q'(|(x,u_1)|)} M'(x, u_1, v_1) = 1$ iff $\exists u_1, v_1 \in \{0,1\}^{q'(|x|)} M'(x, u_1, v_1) = 1$. In other words, M' is a deterministic TM taking x as input and u_1, v_1 as a certificate (Can combine the 2 existential certificates by concatenating the two strings separated by a 1 in between). Thus, $L \in NP$. This implies that Σ_2^p in NP. Similarly, we can show that $\Pi_2^p \in NP$.
   Induction Hypothesis: For all $j < i$, $\Sigma_j^p, \Pi_j^p \subseteq NP$. Using arguments similar to the base case, show that the truth of the statement can be extended to i.

3. To show that $SUCCINCT - SETCOVER \in \Sigma_p^2$, it is sufficient to provide the appropriate certificate definition for the language. Let $(S = \{\phi_1, \phi_2, \dots, \phi_m\}, k)$ be an instance of SUCCINCT-SETCOVER over variables $U = \{u_1, \dots, u_n\}$. Let S_f denote the truth value of S resulting from the assignment $f : U \to \{0,1\}$. Note that a formula ψ is a tautology if for all assignments f , $\psi(f) = 1$. This immediately gives us the following certificate definition:
   $(S, k) \in SUCCINCT - SETCOVER$ if and only if $\exists S' \subseteq [1, m] \forall f : U \to \{0,1\}, |S'| \leq k$ and $(\vee_{i \in S'} \phi_i)_f = 1$,
   thus showing that $SUCCINCT - SETCOVER \in \Sigma_p^2$.